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# CGE model closures in a skeleton world model

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**NOTE**

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## Foreword

If I were to be cheeky, I would say that this paper is intended for CGE modellers – even seasoned modellers – who are well acquainted with the theoretical foundations of such models, who adapt or mimic existing models for their own applications, and... who *think* they master the itinerary from a set of theoretical equations to a GAMS model program. The person I know who best fits this description is myself. So, originally, I started writing this paper to dispell my own doubts and consolidate my own comprehension. And because I don't think I am unique, I suppose this paper might be of interest to some others.

I am also in perpetual doubt regarding the exactness of mathematical formulations, so I tend to detail all derivations. I succumb to my pet weakness abundantly in this paper. However, to avoid burdening the exposition, the more cumbersome demonstrations have been relegated to appendices.

Some might think that much of what is presented here is of byzantine futility. Perhaps. But then, if one finds no errors in his/her creations, it can only be for want of looking hard enough.

I wish to express my gratitude to Bernard Decaluwé and to the reviewer who examined my paper. Both of them have given me very useful suggestions and encouragement. And the reviewer, confronted with a version of the paper that was much less user friendly than this one, has diligently and heroically examined the paper from beginning to end. To both of them, thanks!

Of course, any remaining errors or inadequacies are mine only.



## CGE MODEL CLOSURES IN A SKELETON WORLD MODEL

There is an abundant literature on macroclosures for single-economy models (Sen, 1963; Rattsø, 1982; Dewatripont and Michel, 1983; and Decaluwé *et al.*, 1987, 1988; for a recent comprehensive review, Delpiazzo, 2010). In this paper, however, we are not concerned with closures in single-economy models, but rather with issues that arise in the context of global multinational models, typically, trade models. In addition to model closures, we explore a few related issues that are rarely dealt with explicitly in CGE multinational modelling:

- model homogeneity
- the role of exchange rates
- calibration consistency
- testing for calibration consistency and model homogeneity

To that effect, we develop a highly simplified skeleton model derived from the PEP-w-1 worldwide CGE model (Lemelin *et al.*, 2013)<sup>1</sup>, but which represents the essential structure of several world trade models. The skeleton model is used to examine and illustrate the issues we tackle. Although our concern is with multinational models, some of the points made in this paper also apply *mutatis mutandis* to single-economy models.

In Section 1, we present the basic concepts: model homogeneity, model closure, and calibration consistency. The theoretical model is presented in Section 2 (Model 1). In Section 3, the model is simplified, and redundant equations are identified and deleted (Model 2). Next (Section 4), the role of exchange rates is clarified, and the model is re-written in terms of the international currency, without exchange rate variables (Model 3). Closure rules are first discussed in reference to Model 3. In Section 5, exchange rates are re-introduced (Model 2 again); closure rules are revisited, detailing the choice between fixed exchange rates (FE) or fixed regional price (FP) closures; and calibration consistency is discussed. The implementation of Model 2 in GAMS is presented in Section 6. A brief conclusion wraps up the paper.

## 1. Concepts

### 1.1 MODEL HOMOGENEITY AND WALRAS' LAW

Formally, a CGE model is a set of simultaneous equations relating variables, some of which are endogenous (determined within the model), the rest being exogenous. The core of a CGE

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<sup>1</sup> <http://www.pep-net.org/pep-standard-cge-models/>

model consists of equations representing consumer and producer optimizing behavior, and market equilibrium. A model solution is a Walrasian competitive general equilibrium: all optimizing economic agents meet their (first-order) optimality conditions, subject to their budget constraints, and all markets are in equilibrium. Without money, the set of equations which constitute the model is homogenous of degree zero in prices.

To formalize the definition of homogeneity and generalize it a little, let us distinguish the three types of variables a model may contain: volume variables, price variables and nominal variables. Some nominal variables are the product of a volume and a price, but others cannot be factored into volume and price; they just represent payments made from one agent to another, such as transfers. With that distinction in mind, a model solution may be characterized as a triplet of vectors  $\{\mathbf{q}, \mathbf{p}, \mathbf{n}\}$ , containing volume ( $\mathbf{q}$ ), price ( $\mathbf{p}$ ) and nominal ( $\mathbf{n}$ ) variables. In a homogenous model, if  $\{\mathbf{q}, \mathbf{p}, \mathbf{n}\}$  is a solution to the model, then  $\{\mathbf{q}, \lambda\mathbf{p}, \lambda\mathbf{n}\}$  is also a solution, for any  $\lambda > 0$ . In other words, multiplying all prices and nominal values by a constant doesn't disturb the equilibrium, because it leaves *relative* prices unchanged. This is the principal implication of homogeneity: only *relative* prices matter. Prices are determined only up to a factor of proportionality and their absolute level is indeterminate.

The absolute level of prices is indeterminate because model homogeneity implies that one equation is redundant: there is one more price variable than the number of independent market equilibrium equations. This is generally referred to as Walras' Law (Léon Walras, 1834-1910). Consider an economy where producers maximize their profits subject to their production function, and consumers own all factors of production, receive all factor income, and maximize their utility subject to their budget constraint<sup>2</sup>. For every good (factor or commodity), excess demand is defined as the sum of demands by all agents, minus the sum of supplies by all agents, for some price vector; denote the vector of excess demands as  $\xi(\mathbf{p})$ . In equilibrium, demand equals supply, and  $\xi(\mathbf{p}^*) = \mathbf{0}$ , where  $\mathbf{p}^*$  is the equilibrium price vector. Walras' Law states that for any  $\mathbf{p}$  (equilibrium or not), the total value of excess demands is zero:  $\mathbf{p}'\xi(\mathbf{p}) = 0$ . This is a straightforward consequence of respecting budget constraints: since  $\mathbf{p}'\xi(\mathbf{p})$  is the sum of all expenditures and incomes of all agents, if all budget constraints are satisfied, then that sum must be zero. A corollary of Walras' Law is that, if all markets but one are in equilibrium for some price vector, then the remaining market must also be in equilibrium, because the value of excess

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<sup>2</sup> This implies that all profits are distributed to consumers. Walras' Law can be demonstrated in a less restrictive setting, but our objective here is to put forth the principle behind Walras' Law, and so we keep its exposition as simple as possible.

demand on that market cannot be different from zero. It follows that in a model with  $N$  markets, there are only  $N-1$  independent market equilibrium equations, and the  $N^{\text{th}}$  is redundant.

That leaves one degree of freedom, and the model is completed by exogenously fixing the price of one good, which plays the role of *numéraire*. The value assigned the price of the numéraire determines the level of prices, and all other prices in the model are expressed in terms of the price of the numéraire<sup>3</sup>.

Homogeneity has two implications regarding the choice of a numéraire and of its value. First, although it may be convenient to set the numéraire at 1, the value assigned the numéraire is arbitrary. That is quite obvious from the definition of homogeneity: if  $\{\mathbf{q}, \mathbf{p}, \mathbf{n}\}$  is a solution to the model when the numéraire is set at 1, then  $\{\mathbf{q}, \lambda\mathbf{p}, \lambda\mathbf{n}\}$  is also a solution, when the numéraire is set at any  $\lambda > 0$ . A second implication is that the choice of numéraire is arbitrary. Suppose that commodity  $i$  is chosen as the numéraire, with its price  $p_i$  set at  $p_i^0$ ; then changing the numéraire for commodity  $j$  and setting its price at  $p_j^1$  is equivalent to changing from solution  $\{\mathbf{q}, \mathbf{p}, \mathbf{n}\}$  to  $\{\mathbf{q}, \lambda\mathbf{p}, \lambda\mathbf{n}\}$  with  $\lambda = p_j^1/p_j^0$ , where  $p_j^0$  is the value of price  $j$  when the numéraire is commodity  $i$ .

In other words, if a model is truly homogenous, the solution values of real (volume) variables and all price and nominal value ratios are supposed to be

- independent of which commodity is taken as the numéraire;
- independent of which region is taken as the reference region when the numéraire is a regional commodity (a particular case of the preceding);
- independent of the particular value given the price of the numéraire, whatever commodity plays that role.

Of course, not all CGE models are “purely” Walrasian. But most non-monetary CGE models nevertheless retain the property of model homogeneity. In any case, if a model is not homogeneous, it should be by purpose, not by accident or by mistake. So it is useful to check for model homogeneity.

It is easy enough to check simple models for homogeneity, but it may be tricky with complex models. With simple models, one can check for homogeneity analytically, by examining the

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<sup>3</sup> Strictly speaking, the word numéraire designates the commodity relative to the price of which the prices of all other commodities are expressed. But for convenience, we take the shortcut of using the word “numéraire” to designate the price of the numéraire commodity. This allows us to say, for example, that all prices are expressed in terms of the numéraire – much more concise and comprehensible than the first sentence of this footnote!

equations. Homogeneity can also be verified numerically, by transforming a  $\{\mathbf{q}, \mathbf{p}, \mathbf{n}\}$  simulation solution into a  $\{\mathbf{q}, \lambda\mathbf{p}, \lambda\mathbf{n}\}$  solution, and then verifying whether the model equations are satisfied. Another approach, somewhat more demanding, is to compare simulation solutions obtained with different values of the numéraire or with different numéraires and verify that the alternate solutions  $\{\mathbf{q}, \mathbf{p}, \mathbf{n}\}$  and  $\{\mathbf{q}', \mathbf{p}', \mathbf{n}'\}$  satisfy the relationship  $\{\mathbf{q}', \mathbf{p}', \mathbf{n}'\} = \{\mathbf{q}, \lambda\mathbf{p}, \lambda\mathbf{n}\}$  for some  $\lambda > 0$ .

The latter approach is more demanding, because it requires that the model equations be written in such a way as to adjust to an arbitrary value of an arbitrary choice of numéraire (this is illustrated in the homogeneity tests outlined in Section 6.2). Moreover, the model closure must be formulated with care to avoid the many pitfalls of inadvertently de-homogenizing the model. Regarding our skeleton model, we shall see, in particular, that, under fixed exogenous current account balance closures, the way current account balances are fixed is critical (Section 4.2).

## 1.2 MODEL CLOSURE

With a CGE model, as with any system of simultaneous equations, the number of independent equations must be equal to the number of endogenous variables for the model to have a solution (the model must be square)<sup>4</sup>. If there are more equations than variables, the model is overdetermined; if there are fewer equations than variables, the model is underdetermined. The issue of model closure concerns the theoretical foundations and meaning of the choices made by the modeler to make the number of equations and the number of variables equal.

In a way, taking into account Walras' Law to discard a redundant equation, and defining a numéraire to make the model square, can be viewed as one aspect of model closure. But the issue of model closure is broader and more substantive.

The discussion of macroclosures in single-economy models was initiated by Sen (1963) in relation to the debate on income distribution. Essentially, Sen showed that competing views of the economy could be characterized by the choice of which equation to eliminate in an overdetermined model (this is nicely summarized in Rattsø, 1983). In contrast, contemporary CGE models are usually underdetermined, left open to different views of the economy. It is up to the model user to choose which equation or constraint to add to the model in order to "close" it and to make it "square", with as many equations as there are variables<sup>5</sup>.

Here we are concerned with multinational (or multiregional) world models, and we will examine

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<sup>4</sup> It would be tempting to add "unique". But the possibility of multiple solutions cannot always be ruled out in models that are constrained nonlinear systems (CNS).

<sup>5</sup> For example, see "Macroeconomic balances", pp. 14-17 in Lofgren *et al.* (2002).

model closures that take the simple form of fixing one or more variables exogenously. Specifically, we consider:

1. How many variables must be exogenously fixed?
2. Which variables can be sensibly designated as exogenous to close the model?
3. Does the choice of exogenous variables always matter? Can the same model closure be implemented by fixing alternative sets of variables? Put otherwise, are there model closures which are different in their implementation, but are mathematically and economically equivalent?

Underlying question 1 is the matter of redundant equations. The number of variables to fix exogenously is equal to the number of degrees of freedom in the model, given by the difference between the number of endogenous variables and the number of independent equations. To accurately count the number of independent equations, one must be able to identify redundant equations in the model. This is of practical importance when the model is submitted to a GAMS solver as a CNS (constrained nonlinear system) class model, because GAMS will reject the model as not square if it contains redundant equations, even if, mathematically, redundant equations do no harm.

### 1.3 CALIBRATION CONSISTENCY

CGE models are routinely calibrated from social accounting matrices (SAMs). SAM entries are transaction flows. Part of the calibration procedure consists in a factoring of SAM transaction flows into price  $\times$  volume products. But SAM transaction flows generally represent composite aggregates for which there is no clear physical unit of measurement. Even when the volume can be measured unambiguously, the factoring is arbitrary: 454 grams of something at 1¢ a gram is the same as one pound at \$4.54 per pound. It follows that the price  $\times$  volume factoring is arbitrary, constrained only by the value of their product. However, although the factoring of SAM transaction flows is arbitrary, there are consistency requirements to satisfy: we return to these shortly.

But first, let us briefly recall an implication of the arbitrariness of price  $\times$  volume factoring that is occasionally neglected in the interpretation of CGE simulation results, namely that prices and volumes must be viewed as *indices*: price or volume levels are meaningless; only proportional *changes* are meaningful. When, in a simulation solution, a transactions flow increases or decreases relative to its benchmark value, the CGE model provides a decomposition of *the change* into price variation and volume variation. This is exactly the kind of decomposition that is needed to correctly interpret, say, an increase in consumption expenditures when the consumer

price index changes.

Regarding consistency requirements, it must be kept in mind that the model links prices and volumes through identities and other relationships. For example, the tax-inclusive price of a commodity and its tax-exclusive price are related by the tax rate. Consequently, although the price  $\times$  volume factoring is arbitrary, it must be done consistently: the arbitrarily assigned values of prices or volumes must be mutually compatible within the model. Whence the necessity of a calibration strategy that, after arbitrarily fixing the values of some set of variables (or, equivalently, imposing arbitrary normalization rules), then deduces the values of other prices and volumes using the model relationships.

In more general terms, model parametrization consists in assigning magnitudes to parameters and benchmark values to variables, on the basis of SAM values and model relationships (equations). Once the so-called “free” parameters (elasticities and the like) have been specified, the number of unknowns (parameters and variables) usually remains greater than the number of constraints (SAM values and model equations). Therefore, to solve the calibration problem, it is necessary to impose additional constraints. There is in general no unique set of additional constraints that will complete the calibration procedure, but such constraints, given that they are based neither on observation (SAM flows), nor on theory (model equations), should be “neutral”, non-restrictive. And if they are not restrictive, they do not affect model results. This is the idea behind calibration consistency.

Formally, we define calibration consistency as follows. A model is said to be calibrated consistently if the relative variation of a variable between any simulation solution and its benchmark is the same, no matter which particular set of additional non-restrictive constraints is imposed to solve the calibration problem. Mathematically, let  $P_{i,k}^S$  be the value of price variable  $i$ , in solution  $k$  of the model calibrated using a set  $S$  of additional constraints; subscript  $k$  is equal to 0 for the benchmark, and to 1 for the simulation. Also let  $Q_{j,k}^S$  be the value of volume variable  $j$  in solution  $k$  of the model calibrated using  $S$ . Sets  $S$  may differ in the choice of variables or in the values assigned or both. Then the calibration procedure is consistent if

$$\frac{P_{i,1}^{S1}}{P_{i,0}^{S1}} = \frac{P_{i,1}^{S2}}{P_{i,0}^{S2}} \quad [001]$$

$$\frac{Q_{j,1}^{S1}}{Q_{j,0}^{S1}} = \frac{Q_{j,1}^{S2}}{Q_{j,0}^{S2}} \quad [002]$$

for any pair  $S1, S2$  of sets of additional calibration constraints.

If conditions [001] and [002] are not realized, it may be for one of two reasons. First, one of the pair of sets of constraints  $S_1$  and  $S_2$ , or both, may be restrictive (non-neutral). In that case, the problem is with the additional constraints, not with the calibration procedure. And either the additional calibration constraints must be replaced with a set of neutral constraints, or the model should be extended to include one or more of these non-neutral constraints. A concrete example of non-restrictiveness is given in subsection 5.2.2 with respect to the arbitrariness of price  $\times$  volume factoring. We show that the volume of a CES or CET aggregate may be benchmarked as the sum of its components without implicitly imposing perfect substitutability, or, alternately, that the price of the aggregate may be arbitrarily set at 1 without imposing price equality. An example of constraints that do impose restrictions on the model is the assignment of values to the elasticity parameters: changing the elasticities modifies the model. Two variants of the same model with different elasticity values are not equivalent. For that reason, modellers try to use econometric estimates of elasticities, or else to borrow estimates from the literature.

The second reason for which conditions [001] and [002] may be violated is that the way in which the constraints are implemented is inconsistent in the sense that the calibration procedure is compatible only with one particular set of additional constraints. This is what we refer to specifically as calibration inconsistency. An example is given in subsection 5.2.3, where a calibration shortcut which sets a given price at 1 is valid only if other, previously determined prices are also set at 1, while their value should be arbitrary.

To summarize, model parameters and benchmark values cannot be calibrated from SAM values and model equations alone: the solution of the calibration problem remains undetermined without additional constraints. However, these additional constraints are not totally arbitrary: they must be non-restrictive and applied consistently.

## 2. Skeleton model: Model 1

We now define the theoretical model which will serve as our departure point to explore the issues raised in the introduction.

### 2.1 MODEL DESCRIPTION

There are  $N$  regions. There is a single regional agent, and there are no taxes.

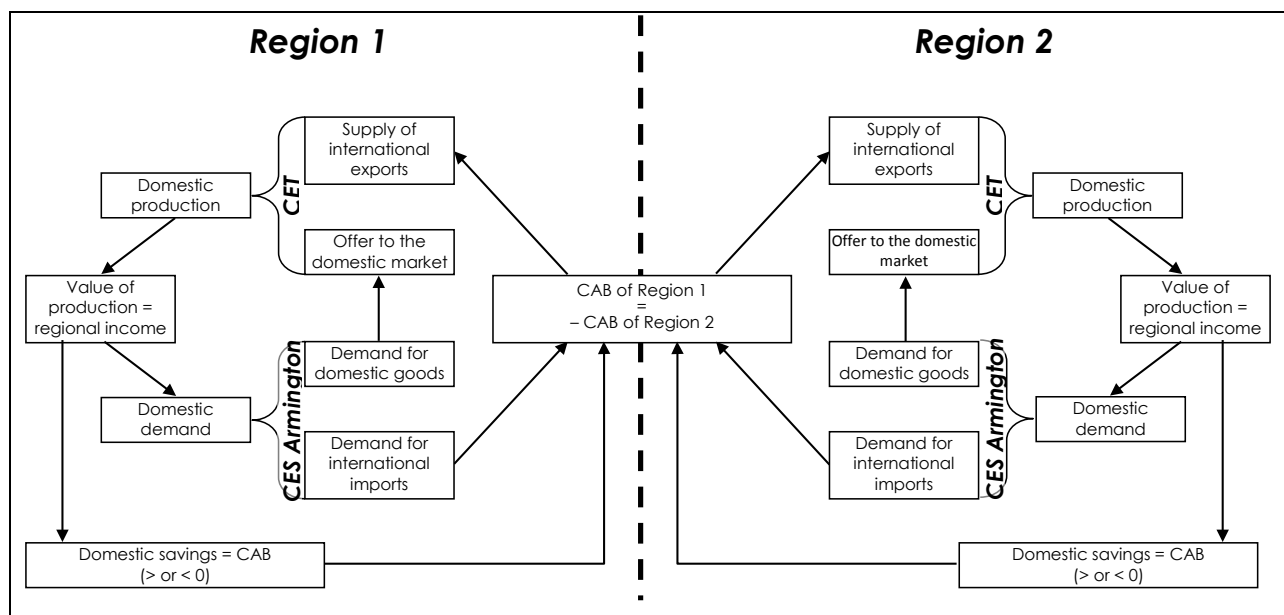
There is a single good, and production factors are fixed. With full employment, this implies that output is fixed in each region  $z$ , and that the regional agent's income is equal to the value of production.

The model is static, so it is unnecessary to distinguish between consumption and investment. It follows that savings, the difference between income and consumption, are equal to the current account balance (CAB).

The model retains the Armington hypothesis regarding the distribution of demand between imports and the domestically produced good, and, for imports, between source regions, using CES aggregator functions. Similarly, production is allocated between domestic sales and exports, and, for exports, between regions of destination, according to CET functions.

This model is close to a Walrasian pure exchange economy. It is illustrated in Figure 1 for the case of two regions. But in what follows, we deal with an  $N$ -region model.

**Figure 1 – Skeleton model of international trade**



At the top of Figure 1, domestic production is sold on the domestic market or exported, following a CET transformation function. The value of domestic production constitutes the regional agent's income, which is divided between domestic demand, and domestic savings (which may be negative). Domestic demand is distributed following a CES function between domestic production and imports. The imperfect-substitutability Armington hypothesis is what makes it possible to have both imports and exports even if there is a single good. Each region's current account balance (CAB) is given by the value of exports, minus the value of imports. Absent investment, the CAB is equal to domestic savings. Also, in the particular case of two regions, one region's CAB is equal to minus the other's.

Table 1 lists the Model 1 variables, and Table 2 lists the equations. The regions are designated by

subscripts<sup>6</sup>  $z, zj, zjj \in \{1, \dots, N\}$ . For instance, in variables  $IM_{zj,z}$ ,  $EX_{z,zj}$  and  $PW_{z,zj}$ , the first subscript designates the region of origin, and the second the region of destination of goods in trade flows. This physical flow from-to convention for subscript order is applied throughout the model. Also note that  $IM_{z,z} = 0$ ,  $EX_{z,z} = 0$ , and  $PW_{z,z}$  is undefined, which reflects the fact that there is no international trade between a country and itself.<sup>7</sup>

The model is presented in more detail in Appendix A. The number of variables in the model is  $12N + 3N(N - 1)$ , where  $N$  is the number of regions. The number of equations in Table 2 is  $12N + 1 + 3N(N - 1)$  equations. The model appears to be overdetermined. But, as we shall see, it is not.

**Table 1 – Model 1 variables**

		<b>Number of variables</b>
<b>Volumes</b>		
$XS_z$	Domestic production in region z	N
$Q_z$	Domestic demand for the composite good in region z	N
$D_z$	Domestic demand for the locally produced good in region z	N
$IM_{zj,z}$	Imports from region zj by region z	N (N-1)
$IMT_z$	Total imports of region z	N
$EX_{z,zj}$	Exports by region z to region zj	N (N-1)
$EXT_z$	Total exports of region z	N
<b>Prices</b>		
$P_z$	Producer price in region z	N
$PL_z$	Market price of local product in region z	N
$PC_z$	Price of the composite good in region z	N
$PMT_z$	Price of composite imports in region z	N
$PXT_z$	Price of composite exports of region z	N
$e_z$	Exchange rate (price of the international currency in terms of region z's currency)	N
$PW_{z,zj}$	World price of exports from region z to region zj	N (N-1)
<b>Nominal value variables</b>		
$CAB_z$	Current account balance of region z	N
<b>Total number of variables</b>		<b>12N + 3N(N-1)</b>

<sup>6</sup> A different style of subscripting might use  $i, j$  and  $k$ , rather than  $z, zj$  and  $zjj$ .

<sup>7</sup> This is in contrast to PEP-w-1, where there is international trade within multi-country regions.

Table 2 – Model 1 equations

	Number of equations
$CAB_z = P_z XS_z - PC_z Q_z$	[003] N
$CAB_z = PXT_z EXT_z - PMT_z IMT_z$	[004] N
$XS_z = B_z \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}}$ where $\kappa_z = \frac{\tau_z + 1}{\tau_z}$ , with $0 < \tau_z < \infty$	[005] N
$\frac{EXT_z}{D_z} = \left( \frac{\beta_z PXT_z}{1 - \beta_z PL_z} \right)^{\tau_z}$	[006] N
$EXT_z = B_z^X \left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}}$ where $\kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}$ , with $0 < \tau_z^X < \infty$	[007] N
$EX_{z,zj} = \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \left[ \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X PXT_z} \right]^{\tau_z^X}$	[008] N (N-1)
$Q_z = A_z \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{\frac{-1}{\rho_z}}$ where $\rho_z = \frac{1 - \sigma_z}{\sigma_z}$ , with $0 < \sigma_z < \infty$	[009] N
$\frac{IMT_z}{D_z} = \left( \frac{1 - \alpha_z PL_z}{\alpha_z PMT_z} \right)^{\sigma_z}$	[010] N
$IMT_z = A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M IM_{zj,z}^{\rho_z^M} \right]^{\frac{-1}{\rho_z^M}}$ where $\rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M}$ , with $0 < \sigma_z^M < \infty$	[011] N
$IM_{zj,z} = \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} \left[ \frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right]^{\sigma_z^M}$	[012] N (N-1)
$P_z XS_z = PL_z D_z + PXT_z EXT_z$	[013] N
$PXT_z EXT_z = e_z \sum_{zj} PW_{z,zj} EX_{z,zj}$	[014] N
$PC_z Q_z = PL_z D_z + PMT_z IMT_z$	[015] N
$PMT_z IMT_z = e_z \sum_{zj} PW_{zj,z} IM_{zj,z}$	[016] N
$EX_{zj,z} = IM_{zj,z}$	[017] N (N-1)
$\sum_z \frac{CAB_z}{e_z} = 0$	[018] 1
<b>Total number of equations</b>	<b>12 N + 3 N (N-1) + 1</b>

## 2.2 HOMOGENEITY OF MODEL 1

Model 1 being a very simple model, it is easy to verify its homogeneity analytically. Indeed, for

any set of variable values that solves equations [003]-[018], a new set of variable values obtained by multiplying all prices and nominal values by some positive constant  $\lambda$  is also a solution, with the proviso that whenever the product  $e_z PW_{z,zj}$  appears, the proportional change be applied either to  $PW_{z,zj}$ , or to  $e_z$ , but not to both. Indeed, observe that in all equations where  $PW_{z,zj}$  appears, it is multiplied by  $e_z$ . The fact is that the product  $e_z PW_{z,zj}$  should be considered to be a single price variable. Further, we shall see below that the exchange rate variables are really superfluous.

The homogeneity of Model 1 is demonstrated in Table 3 below. Visual inspection shows that the  $\lambda$  constants cancel out in every equation. Homogeneity implies that in the model, prices (and nominal values) are defined only up to a factor of proportionality, so that the modeler must choose a numéraire and set its level.

### 3. Redundant equations and reduction of the model: Model 2

From a careful examination of Tables 1 and 2, Model 1 appears to have  $12N + 3N(N - 1)$  variables: 12 (groups of) variables indexed in  $z$ , and 3 (groups of) variables indexed in  $z,zj$ . The equation count is  $12N + 3N(N - 1) + 1$ : 12 (groups of) equations indexed in  $z$ , 3 (groups of) equations indexed in  $z,zj$ , plus the single equation [018]. So the model appears to have one equation too many. However, as we now demonstrate, several equations are redundant, and we now proceed to clean up the model, discarding redundant equations.

#### 3.1 REDUNDANT EQUATIONS

##### 3.1.1 Equation [014] redundant

Given [\[007\]](#) and [\[008\]](#), [equation \[014\]](#) is redundant. The demonstration is somewhat cumbersome, so it is given in Appendix B.

##### 3.1.2 Equation [016] redundant

Given [\[011\]](#) and [\[012\]](#), [equation \[016\]](#) is redundant. The demonstration is somewhat cumbersome, so it is given in Appendix B, parallel to the preceding.

**Table 3 – Homogeneity of Model 1**

$\lambda CAB_z = \lambda P_z XS_z - \lambda PC_z Q_z$	[003]
$\lambda CAB_z = \lambda PXT_z EXT_z - \lambda PMT_z IMT_z$	[004]
$XS_z = B_z [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z}}$ where $\kappa_z = \frac{\tau_z + 1}{\tau_z}$ , with $0 < \tau_z < \infty$	[005]
$\frac{EXT_z}{D_z} = \left( \frac{\beta_z \lambda PXT_z}{1 - \beta_z \lambda PL_z} \right)^{\tau_z}$	[006]
$EXT_z = B_z^X \left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}}$ where $\kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}$ , with $0 < \tau_z^X < \infty$	[007]
$EX_{z,zj} = \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \left[ \frac{\lambda e_z PW_{z,zj}}{\beta_{z,zj}^X \lambda PXT_z} \right]^{\tau_z^X}$	[008]
$Q_z = A_z [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{\frac{1}{\rho_z}}$ where $\rho_z = \frac{1 - \sigma_z}{\sigma_z}$ , with $0 < \sigma_z < \infty$	[009]
$\frac{IMT_z}{D_z} = \left( \frac{1 - \alpha_z \lambda PL_z}{\alpha_z \lambda PMT_z} \right)^{\sigma_z}$	[010]
$IMT_z = A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}}$ where $\rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M}$ , with $0 < \sigma_z^M < \infty$	[011]
$IM_{zj,z} = \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} \left[ \frac{\alpha_{zj,z}^M \lambda PMT_z}{\lambda e_z PW_{zj,z}} \right]^{\sigma_z^M}$	[012]
$\lambda P_z XS_z = \lambda PL_z D_z + \lambda PXT_z EXT_z$	[013]
$\lambda PXT_z EXT_z = e_z \sum_{zj} \lambda PW_{z,zj} EX_{z,zj}$	[014]
$\lambda PC_z Q_z = \lambda PL_z D_z + \lambda PMT_z IMT_z$	[015]
$\lambda PMT_z IMT_z = e_z \sum_{zj} \lambda PW_{zj,z} IM_{zj,z}$	[016]
$EX_{zj,z} = IM_{zj,z}$	[017]
$\sum_z \frac{\lambda CAB_z}{\lambda e_z} = 0$	[018]

**3.1.3 Equation [018] redundant**

Divide [004] throughout by  $e_z$ :

$$\frac{CAB_z}{e_z} = \sum_{zj} PW_{z,zj} EX_{z,zj} - \sum_{zj} PW_{zj,z} IM_{zj,z} \quad [019]$$

Sum over z.

$$\sum_z \frac{CAB_z}{e_z} = \sum_z \sum_{zj} PW_{z,zj} EX_{z,zj} - \sum_z \sum_{zj} PW_{zj,z} IM_{zj,z} \quad [020]$$

$$\sum_z \frac{CAB_z}{e_z} = \sum_z \sum_{zj} PW_{z,zj} EX_{z,zj} - \sum_z \sum_{zj} PW_{z,zj} IM_{z,zj} \quad [021]$$

$$\sum_z \frac{CAB_z}{e_z} = \sum_z \sum_{zj} PW_{z,zj} (EX_{z,zj} - IM_{z,zj}) \quad [022]$$

And, given [\[017\]](#) we have [\[018\]](#).

Equations [004] and [017] together imply [018], which is therefore redundant.

### 3.1.4 Equation [004] redundant

Substitute [\[013\]](#) and [\[015\]](#) into [\[003\]](#) to obtain [\[004\]](#).

Equations [013], [015] and [003] together imply [004], which is therefore redundant.

### 3.1.5 Walras' Law

We shall now examine how Walras' Law applies to our model. Define excess demands on the domestic and international markets respectively as

$$XD_z = D_z^D - D_z^O \quad [023]$$

$$XM_{zj,z} = IM_{zj,z} - EX_{zj,z} \quad [024]$$

where

$$D_z^D = (A_z)^{\sigma_z - 1} \left( \frac{\alpha_z PC_z}{PL_z} \right)^{\sigma_z} Q_z \quad [025]$$

$$D_z^O = \left( \frac{1}{B_z} \right)^{\tau_z + 1} \left( \frac{PL_z}{\beta_z P_z} \right)^{\tau_z} XS_z \quad [026]$$

$$IM_{zjj,z} = \left( \frac{1}{A_z^M} \right)^{1 - \sigma_z^M} \left( \frac{PW_{zjj,z}}{\alpha_{zjj,z}^M PMT_z} \right)^{-\sigma_z^M} (A_z)^{\sigma_z - 1} \left( \frac{(1 - \alpha_z) PC_z}{PMT_z} \right)^{\sigma_z} Q_z \quad [027]$$

$$EX_{z,zjj} = \left( \frac{1}{B_z^X} \right)^{\tau_z^X + 1} \left( \frac{PW_{z,zjj}}{\beta_{z,zjj}^X PXT_z} \right)^{\tau_z^X} \left( \frac{1}{B_z} \right)^{\tau_z + 1} \left( \frac{PXT_z}{(1 - \beta_z) P_z} \right)^{\tau_z} XS_z \quad [028]$$

Supply and demand equations [025]-[028] are homogeneous with respect to prices. Their mathematical derivations are given in Appendices C-F.

The value of excess demand on the domestic and international markets is given by

$$PL_z X D_z = PL_z D_z^D - PL_z D_z^O \quad [029]$$

$$PW_{zj,z} X M_{zj,z} = PW_{zj,z} I M_{zj,z} - PW_{zj,z} E X_{zj,z} \quad [030]$$

The aggregate value of all excess demands, expressed in terms of the international currency, is

$$\begin{aligned} \sum_z \frac{PL_z}{e_z} X D_z + \sum_z \sum_{zj} PW_{z,zj} X M_{z,zj} = \\ \sum_z \frac{PL_z}{e_z} D_z^D - \sum_z \frac{PL_z}{e_z} D_z^O + \sum_z \sum_{zj} PW_{z,zj} I M_{z,zj} - \sum_z \sum_{zj} PW_{z,zj} E X_{z,zj} \end{aligned} \quad [031]$$

Now, equations [\[014\]](#) and [\[016\]](#) are no longer present among the model equations (they have already been discarded as redundant), but they are implied<sup>8</sup>:

- equations [007] and [008] together imply [014] (see above and Appendix B);
- equations [011] and [012] together imply [016] (see above and Appendix B).

Substitute equations [014] and [016] into [031] to obtain

$$\begin{aligned} \sum_z \frac{PL_z}{e_z} X D_z + \sum_z \sum_{zj} PW_{z,zj} X M_{z,zj} = \\ \sum_z \frac{PL_z}{e_z} D_z^D - \sum_z \frac{PL_z}{e_z} D_z^O + \sum_{zj} \frac{PMT_{zj}}{e_{zj}} I M T_{zj} - \sum_z \frac{PXT_z}{e_z} E X T_z \end{aligned} \quad [032]$$

$$\begin{aligned} \sum_z \frac{PL_z}{e_z} X D_z + \sum_z \sum_{zj} PW_{z,zj} X M_{z,zj} = \\ \sum_z \frac{PL_z}{e_z} D_z^D - \sum_z \frac{PL_z}{e_z} D_z^O + \sum_z \frac{PMT_z}{e_z} I M T_z - \sum_z \frac{PXT_z}{e_z} E X T_z \end{aligned} \quad [033]$$

$$\begin{aligned} \sum_z \frac{PL_z}{e_z} X D_z + \sum_z \sum_{zj} PW_{z,zj} X M_{z,zj} = \\ \sum_z \left[ \left( \frac{PL_z}{e_z} D_z^D + \frac{PMT_z}{e_z} I M T_z \right) - \left( \sum_z \frac{PL_z}{e_z} D_z^O + \sum_z \frac{PXT_z}{e_z} E X T_z \right) \right] \end{aligned} \quad [034]$$

---

<sup>8</sup> We underline this as a precaution against the logical pitfall that would consist in using a discarded equation that could no longer be considered implicit in the model because the equations which imply it would also be absent from the model.

$$\sum_z \frac{PL_z}{e_z} XD_z + \sum_z \sum_{zj} PW_{z,zj} XM_{z,zj} = \sum_z \frac{1}{e_z} \left[ (PL_z D_z^D + PMT_z IMT_z) - \left( \sum_z PL_z D_z^O + \sum_z PXT_z EXT_z \right) \right] \quad [035]$$

Substitute [\[013\]](#) and [\[015\]](#) into [035], and

$$\sum_z \frac{PL_z}{e_z} XD_z + \sum_z \sum_{zj} PW_{z,zj} XM_{z,zj} = \sum_z \frac{1}{e_z} (PC_z Q_z - P_z XS_z) \quad [036]$$

where the regional agents' budget constraints are given by [\[003\]](#).

Regional budget constraints are homogeneous with respect to prices and nominal values<sup>9</sup>.

Substitute the regional budget constraints into [036] to obtain

$$\sum_z \frac{PL_z}{e_z} XD_z + \sum_z \sum_{zj} PW_{z,zj} XM_{z,zj} = \sum_z \frac{1}{e_z} (-CAB_z) \quad [037]$$

Equation [\[018\]](#) is no longer present among the model equations, but implied: equations [\[013\]](#), [\[015\]](#) and [\[003\]](#) together imply [\[004\]](#), and equations [\[004\]](#) and [\[017\]](#) together imply [\[018\]](#).

Substitute [\[018\]](#) into [037] to obtain

$$\sum_z \frac{PL_z}{e_z} XD_z + \sum_z \sum_{zj} PW_{z,zj} XM_{z,zj} = 0 \quad [038]$$

So the total value of excess demands is zero, which is Walras' Law. It follows from Walras' Law that, if all excess demands but one are zero, then the remaining one is zero also. Therefore, one of the market equilibrium constraints consisting of [\[017\]](#) and

$$D_z^D = D_z^O \quad [039]^{10}$$

is redundant.

However, rather than discarding one of the market equilibrium constraints, it is possible to discard one of the budget constraints, while retaining all market equilibrium constraints. Indeed, if all excess demands are zero for some price vector, the left-hand side of equation [036] is zero.

---

<sup>9</sup> In the traditional development of Walras' Law, the income and expenditures of each agent are constrained to be equal, so that their budget constraints are homogeneous with respect to prices. Here, agents may have surpluses or deficits (non-zero CABs), the counterpart of which are international loans, broadly speaking. It follows that homogeneity must be defined not with respect to prices only, but with respect to prices *and* nominal values.

<sup>10</sup> Of course, in the model, this equilibrium constraint is represented by the fact that both sides of the equation are one and the same variable,  $D_z$ .

It follows that, for any region  $z$ ,  $z \in \{1, \dots, N\}$ ,

$$\frac{1}{e_z} (PC_z Q_z - P_z X S_z) = - \sum_{zj \neq z} \frac{1}{e_{zj}} (PC_{zj} Q_{zj} - P_{zj} X S_{zj}) \quad [040]$$

while [018] implies

$$\frac{CAB_z}{e_z} = - \sum_{zj \neq z} \frac{CAB_{zj}}{e_{zj}} \quad [041]$$

Consequently, if

$$CAB_{zj} = P_{zj} X S_{zj} - PC_{zj} Q_{zj}, \text{ for all } zj \neq z \quad [042]^{11}$$

then [040] and [041] guarantee that, for the remaining region  $z$  also,

$$CAB_z = P_z X S_z - PC_z Q_z \quad [043]$$

This is the form which Walras' Law takes in our model. We arbitrarily pick some region  $z_{leon}$ ,  $z_{leon} \in \{1, \dots, N\}$  ( $z_{leon}$  is a mnemonic for Léon Walras), and remove equation [043] for that single region. Note that, with the removal of the equation relating to  $z_{leon}$ , the variable  $CAB_{z_{leon}}$  no longer appears in the model. Its value may be computed using the suppressed equation.

It is common practice in CGE modelling to introduce an extra variable and an extra equation to verify Walras' Law. In the GAMS implementation described in 6.1, the extra variable is labeled  $LEON$  in honor of Léon Walras, and  $LEON = CAB_{z_{leon}} - (P_{z_{leon}} X S_{z_{leon}} - PC_{z_{leon}} Q_{z_{leon}})$ . A nonzero  $LEON$  in the solution indicates that there is an error in the model.

### 3.1.6 Summary

We have shown that

- equations [007] and [008] together imply [014], which is therefore redundant;
- equations [011] and [012] together imply [016], which is therefore redundant;
- equations [004] and [017] together imply [018], which is therefore redundant;
- equations [013], [015] and [003] together imply [004], which is therefore redundant;
- equations
  - [014] (or equivalently the combination of [007] and [008]),

---

<sup>11</sup> Equation [042] is identical to [003], except that it is restricted to  $z \neq z_{leon}$ .

- [016] (or equivalently the combination of [011] and [012]),
- [013], [015] and [003],
- [018] (or alternatively the combination of [013], [015] and [003], which together imply [004], and equations [004] and [017])

together imply that, if equation [003] is satisfied for  $N - 1$  regions, then it is also satisfied for the  $N^{\text{th}}$  one (Walras' Law). Therefore, one equation of the set [003] may be discarded as redundant.

### 3.2 MODEL 2

After eliminating [004], [014], [016] [018], and one equation of the set [003], we are left with Model 2. The equations in Model 2, listed in Table 4, number  $9N - 1 + 3N(N - 1)$ , where  $N$  is the number of regions. The list of variables is given in Table 5.  $CAB_{zleon}$  no longer appears in the model; it is determined implicitly. Since it no longer appears in the model,  $CAB_{zleon}$  is not to be counted as a model variable, even if it figures in Table 4. So the reformulated model has  $12N - 1 + 3N(N - 1)$  variables.

Thus, Model 2 is underdetermined, it is  $3N$  equations short. So the overdetermined appearance of Model 1 is nothing more than an illusion caused by the presence of redundant equations. Consequently, the closure of Model 2 will require adding  $3N$  constraints, including the numéraire.

We also note that the homogeneity of Model 2 can be confirmed by examining it analytically in the same way as Model 1.

**Table 4 – Model 2 equations**

$$CAB_z = P_z XS_z - PC_z Q_z, \text{ for all } z \neq z_{\text{leon}} \quad [042]$$

$$XS_z = B_z \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} \text{ where } \kappa_z = \frac{\tau_z + 1}{\tau_z}, \text{ with } 0 < \tau_z < \infty \quad [005]$$

$$\frac{EXT_z}{D_z} = \frac{\left( \frac{\beta_z}{PL_z} \right)^{\tau_z}}{\left( \frac{1 - \beta_z}{PXT_z} \right)^{\tau_z}} = \left( \frac{\beta_z}{1 - \beta_z} \frac{PXT_z}{PL_z} \right)^{\tau_z} \quad [006]$$

$$EXT_z = B_z^X \left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} \text{ where } \kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}, \text{ with } 0 < \tau_z^X < \infty \quad [007]$$

$$EX_{z,zj} = \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \left[ \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X PXT_z} \right]^{\tau_z^X} \quad [008]$$

$$Q_z = A_z \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{\frac{-1}{\rho_z}} \text{ where } \rho_z = \frac{1 - \sigma_z}{\sigma_z}, \text{ with } 0 < \sigma_z < \infty \quad [009]$$

$$\frac{IMT_z}{D_z} = \frac{\left( \frac{1 - \alpha_z}{PMT_z} \right)^{\sigma_z}}{\left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z}} = \left( \frac{1 - \alpha_z}{\alpha_z} \frac{PL_z}{PMT_z} \right)^{\sigma_z} \quad [010]$$

$$IMT_z = A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} \text{ where } \rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M}, \text{ with } 0 < \sigma_z^M < \infty \quad [011]$$

$$IM_{zj,z} = \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} \left[ \frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right]^{\sigma_z^M} \quad [012]$$

$$P_z XS_z = PL_z D_z + PXT_z EXT_z \quad [013]$$

$$PC_z Q_z = PL_z D_z + PMT_z IMT_z \quad [015]$$

$$EX_{zj,z} = IM_{zj,z} \quad [017]$$

**Table 5 – Model 2 variables**

<b>Volumes</b>	
$XS_z$	Domestic production in region z
$Q_z$	Domestic demand for the composite good in region z
$D_z$	Domestic demand for the locally produced good in region z
$IM_{zj,z}$	Imports from region zj by region z
$IMT_z$	Total imports of region z
$EX_{z,zj}$	Exports by region z to region zj
$EXT_z$	Total exports of region z
<b>Prices</b>	
$P_z$	Producer price in region z
$PL_z$	Market price of local product in region z
$PC_z$	Price of the composite good in region z
$PMT_z$	Price of composite imports in region z
$PXT_z$	Price of composite exports of region z
$e_z$	Exchange rate (price of the international currency in terms of region z's currency)
$PW_{z,zj}$	World price of exports from region z to region zj
<b>Nominal value variables</b>	
$CAB_z$	Current account balance of region z

## 4. Model in terms of the international currency: Model 3

Before proceeding to discussing Model 2 closure, it will prove enlightening to rewrite the model in terms of the international currency.

### 4.1 VARIABLE TRANSFORMATION

Since  $e_z$  is the price of the international currency in terms of region z's domestic currency, all regional prices and nominal values in the model can be expressed in terms of the international currency by dividing them by  $e_z$ . So let

$$P_z^* = P_z/e_z \quad [044]$$

$$PL_z^* = PL_z/e_z \quad [045]$$

$$PC_z^* = PC_z/e_z \quad [046]$$

$$PMT_z^* = PMT_z/e_z \quad [047]$$

$$PXT_z^* = PXT_z/e_z \quad [048]$$

$$CAB_z^* = CAB_z/e_z \quad [049]$$

With these substitutions, we write Model 3: Table 6 gives the list of Model 3 equations, and Table 7 its variables. Since Model 3 equations are equivalent to Model 2 equations, their identification numbers have not been changed.

**Table 6 – Model 3 equations**

$CAB_z^* = P_z^* X S_z - PC_z^* Q_z, z \neq z_{leon}$	[042]
$X S_z = B_z [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z}}$ where $\kappa_z = \frac{\tau_z + 1}{\tau_z}$ , with $0 < \tau_z < \infty$	[005]
$\frac{EXT_z}{D_z} = \frac{\left(\frac{\beta_z}{PL_z^*}\right)^{\tau_z}}{\left(\frac{1 - \beta_z}{PXT_z^*}\right)^{\tau_z}} = \left(\frac{\beta_z}{1 - \beta_z} \frac{PXT_z^*}{PL_z^*}\right)^{\tau_z}$	[006]
$EXT_z = B_z^X \left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}}$ where $\kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}$ , with $0 < \tau_z^X < \infty$	[007]
$EX_{z,zj} = \frac{EXT_z}{(B_z^X)^{1 + \tau_z^X}} \left[ \frac{PW_{z,zj}}{\beta_{z,zj}^X PXT_z^*} \right]^{\tau_z^X}$	[008]
$Q_z = A_z [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{\frac{-1}{\rho_z}}$ where $\rho_z = \frac{1 - \sigma_z}{\sigma_z}$ , with $0 < \sigma_z < \infty$	[009]
$\frac{IMT_z}{D_z} = \frac{\left(\frac{1 - \alpha_z}{PMT_z^*}\right)^{\sigma_z}}{\left(\frac{\alpha_z}{PL_z^*}\right)^{\sigma_z}} = \left(\frac{1 - \alpha_z}{\alpha_z} \frac{PL_z^*}{PMT_z^*}\right)^{\sigma_z}$	[010]
$IMT_z = A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}}$ where $\rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M}$ , with $0 < \sigma_z^M < \infty$	[011]
$IM_{zj,z} = \frac{IMT_z}{(A_z^M)^{1 - \sigma_z^M}} \left[ \frac{\alpha_{zj,z}^M PMT_z^*}{PW_{zj,z}} \right]^{\sigma_z^M}$	[012]
$P_z^* X S_z = PL_z^* D_z + PXT_z^* EXT_z$	[013]
$PC_z^* Q_z = PL_z^* D_z + PMT_z^* IMT_z$	[015]

**Table 7 – Model 3 variables**

<b>Volumes</b>	
$XS_z$	Domestic production in region z
$Q_z$	Domestic demand for the composite good in region z
$D_z$	Domestic demand for the locally produced good in region z
$IM_{zj,z}$	Imports from region zj by region z
$IMT_z$	Total imports of region z
$EX_{z,zj}$	Exports by region z to region zj
$EXT_z$	Total exports of region z
<b>Prices</b>	
$P_z^*$	Producer price
$PL_z^*$	Market price of local product
$PC_z^*$	Price of the composite good
$PMT_z^*$	Price of composite imports to region z
$PXT_z^*$	Price of composite exports from region z
$PW_{z,zj}$	World price of exports from region z to region zj
<b>Nominal value variables</b>	
$CAB_z^*$	Current account balance

Homogeneity can be demonstrated analytically for Model 3, in the same way as for the two preceding models.

By construction, Model 3 is exactly equivalent to Model 2, since it has been obtained by a transformation of variables. Conversely, any solution of Model 3 can be converted to a solution of Model 2 by applying the following reverse conversions:

$$P_z = e_z P_z^* \quad [050]$$

$$PL_z = e_z PL_z^* \quad [051]$$

$$PC_z = e_z PC_z^* \quad [052]$$

$$PMT_z = e_z PMT_z^* \quad [053]$$

$$PXT_z = e_z PXT_z^* \quad [054]$$

$$CAB_z = e_z CAB_z^* \quad [055]$$

which are simply inversions of [\[044\] to \[049\]](#). This holds for any set of  $e_z > 0$ , where the values of  $e_z$  need not be equal. We conclude that the exchange rates in Model 2 are totally arbitrary, or else that they are superfluous, as demonstrated in Model 3. Indeed, several world trade models, such as MIRAGE (Bchir *et al.*, 2002a, 2002b; Decreux and Valin, 2007), are expressed in terms of

an international currency and have no exchange rates.

In Model 3 however, exchange rates  $e_z$  no longer appear in the equations. Hence, there are  $N$  variables less than in Model 2, but the same number of equations. Specifically, Model 3 has  $9N - 1 + 3N(N - 1)$  equations, like Model 2, and  $11N - 1 + 3N(N - 1)$  variables. There are  $2N$  degrees of freedom left. We now turn to the issue of closing the model to equate the number of equations and variables.

## 4.2 CLOSURE

Reviewing the list of variables in Table 6, it might appear that there are several candidates for being fixed in the closure rule. Here we shall go through the list critically, discarding possibilities that are incorrect in the general equilibrium framework. Throughout the review process, however, we maintain the neoclassical full-employment hypothesis<sup>12</sup>: with production factors in fixed supply, in our simple model, the neoclassical full-employment hypothesis implies that production is fixed:

$$XS_z = \overline{XS_z} \quad [056]$$

This adds  $N$  constraints to the model, leaving  $N$  degrees of freedom.

### 4.2.1 Neoclassical closure with fixed current account balances

We begin with the closure rule that is probably the most common in trade models, which consists in fixing  $CAB_z^*$  for  $z \neq z_{leon}$ , in addition to fixing the numéraire. This adds  $(N - 1) + 1 = N$  constraints to the model, making it square. However, this is **not** correct in general (although it may be correct in practice under some particular model specifications). The reason is that  $CAB_z^*$  is a nominal variable, and if  $CAB_z^*$  is fixed, the constraint it represents in real terms depends on the choice of numéraire and its value. In other words, if  $CAB_z^*$  is fixed, the model is no longer homogeneous.

Formally, fixing  $CAB_z^*$ ,  $z \neq z_{leon}$ , means fixing  $N - 1$  elements of the vector of nominal variables ( $\mathbf{n}$ ) in the  $\{\mathbf{q}, \mathbf{p}, \mathbf{n}\}$  triplet which represents a model solution (see 1.1 above). Now, homogeneity requires that if two simulations starting from the same set of benchmark values  $\{\bar{\mathbf{q}}, \bar{\mathbf{p}}, \bar{\mathbf{n}}\}$ , with different numéraires, yield solutions  $\{\mathbf{q}, \mathbf{p}, \mathbf{n}\}$  and  $\{\mathbf{q}', \mathbf{p}', \mathbf{n}'\}$  respectively, then the

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<sup>12</sup> Consequently, our discussion of closures is not complete, since the full-employment hypothesis itself is a closure rule.

solutions must satisfy the relationship  $\{\mathbf{q}', \mathbf{p}', \mathbf{n}'\} = \{\mathbf{q}, \lambda \mathbf{p}, \lambda \mathbf{n}\}$  for some  $\lambda > 0$ . If  $N - 1$  elements of the vector of nominal variables ( $\mathbf{n}$ ) are fixed, the relationship can hold only for  $\lambda = 1$ .

Therefore, in a correct closure that preserves model homogeneity, the nominal  $CAB_z^*$  must be replaced with a pseudo-volume variable, or, in other words, current account balances must be fixed in real terms. One way to do this is to fix

$$CABX_z = \frac{CAB_z^*}{PWINDEX} = \frac{CAB_z}{e_z PWINDEX} \quad [057]$$

where

$$PWINDEX = \sqrt{\frac{\sum_z \sum_{zj} PW_{z,zj} EX_{z,zj}^O}{\sum_z \sum_{zj} PW_{z,zj}^O EX_{z,zj}^O} \frac{\sum_z \sum_{zj} PW_{z,zj} EX_{z,zj}}{\sum_z \sum_{zj} PW_{z,zj}^O EX_{z,zj}}} \quad [058]$$

or equivalently, substituting  $IM_{z,zj}$  for  $EX_{z,zj}$ ,

$$PWINDEX = \sqrt{\frac{\sum_z \sum_{zj} PW_{z,zj} IM_{z,zj}^O}{\sum_z \sum_{zj} PW_{z,zj}^O IM_{z,zj}^O} \frac{\sum_z \sum_{zj} PW_{z,zj} IM_{z,zj}}{\sum_z \sum_{zj} PW_{z,zj}^O IM_{z,zj}}} \quad [059]$$

is a Fisher index of bilateral trade prices (superscript O designates benchmark values).

The introduction of  $CABX_z$  and equation [057] adds  $N + 1$  variables, because it re-introduces  $CAB_{zleon}$  into the model, and  $N$  equations. The introduction of  $PWINDEX$  adds one variable and one equation to the model. So the expanded model now has  $N + 1$  additional equations, for a total of  $10N + 3N(N - 1)$  equations, and  $N + 2$  additional variables, for a total of  $12N + 1 + 3N(N - 1)$  variables, leaving  $2N + 1$  degrees of freedom. With  $XS_z$  and  $CABX_z$  fixed ( $2N$  variables), there is a single degree of freedom left. It is used to set the numéraire.

The user may pick any one price from one of the sets  $P_z^*$ ,  $PL_z^*$ ,  $PC_z^*$ ,  $PMT_z^*$ ,  $PXT_z^*$ , or  $PW_{z,zj}$ , or  $PWINDEX$ . Running a simulation with different numéraires will yield solutions where the volume variables are equal, and prices and nominal value variables all proportional (relative prices are the same).

There are of course other ways to define  $CABX_z$ . Among the alternatives is the possibility of using another trade price index, such as a Laspeyres index of bilateral trade prices

$$PWINDEX = \frac{\sum_z \sum_{zj} PW_{z,zj} EX_{z,zj}^O}{\sum_z \sum_{zj} PW_{z,zj}^O EX_{z,zj}^O} = \frac{\sum_z \sum_{zj} PW_{z,zj} IM_{z,zj}^O}{\sum_z \sum_{zj} PW_{z,zj}^O IM_{z,zj}^O} \quad [060]$$

or a Laspeyres index of regions' aggregate import prices

$$PWINDEX = \frac{\sum_z PMT_z^* IMT_z^O}{\sum_z PMT_z^O IMT_z^O} \quad [061]$$

or a Laspeyres index of regions' aggregate export prices

$$PWINDEX = \frac{\sum_z PXT_z^* EXT_z^O}{\sum_z PXT_z^O EXT_z^O} \quad [062]$$

Paasche or any other type of price indexes may also be used. One may even define the “real” current account balance on the basis of a price that is not directly related to trade, such as  $P_{zr}^*$ , the producer price in reference region  $zr \in \{1, \dots, N\}$ <sup>13</sup>:

$$CABX_z = \frac{CAB_z^*}{P_{zr}^*} = \frac{CAB_z}{P_{zr}} \quad [063]$$

It must be kept in mind, however, that the denominator that defines  $CABX_z$  must be the same for all regions  $z$ . It would be an error, for example, to define

$$CABX_z = \frac{CAB_z^*}{P_z^*} = \frac{CAB_z}{P_z} \quad [064]$$

where the denominator is specific to each region. In that case, the implicit solution value of  $CABX_{zleon}$  will be different from its fixed closure value. Since  $CABX_{zleon}$  does not appear in the model, GAMS will find a solution. But the equation left implicit on account of Walras' Law will be violated, as well as one or more of the redundant equations [\[004\]](#), [\[014\]](#), [\[016\]](#), and [\[018\]](#).

No matter how  $CABX_z$  is defined, provided that definition is admissible, the choice of numéraire is arbitrary. However, the way  $CABX_z$  is defined does matter! That is because different variants of  $CABX_z$ , when they are fixed in the closure, impose different constraints in

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<sup>13</sup> The index used for the reference region is  $zr$ , to indicate that the reference region may be different from region  $zleon$  which is excluded from equation [042] under Walras' Law.

real terms. Indeed,  $CABX_z$  is a pseudo-volume variable, which does not have a clear-cut, unique definition: although choosing a particular variant of  $CABX_z$  is *not* equivalent to choosing any other, none of the possible specifications of  $CABX_z$  stands out as the natural one.

Note that the issue of defining  $CABX_z$  is often sidestepped by fixing  $CAB_z$  rather than  $CABX_z$ , an approach we have seen renders the model non-homogenous. When the numéraire is set to 1, fixing  $CAB_z$  rather than  $CABX_z$  is equivalent to defining  $CABX_z$  implicitly as

$$CABX_z = \frac{CAB_z^*}{numeraire^*} \quad [065]$$

For example, if the chosen numéraire is  $PWINDEX$  and it is set to 1, then fixing  $CAB_z$  is equivalent to fixing  $CABX_z$  as defined in [057] above. In other words, when  $CAB_z$ , rather than  $CABX_z$ , is fixed in the closure, the reason why the model is not homogenous is that changing the numéraire implicitly changes the definition of  $CABX_z$ , resulting in a different model.

#### 4.2.2 Neoclassical closure with a set of fixed volume variables

In [Table 7](#), the model is seen to have 6 sets of volume variables, in addition to  $XS_z$ . Fixing  $\overline{XS_z}$  is the form that the neoclassical full-employment hypothesis takes in the skeleton model. We now assert that, if  $\overline{XS_z}$  is fixed, it is infeasible to fix another set of volume variables. This can be illustrated in a 2-region version of Model 3; the [two-region version of Model 3](#) is detailed in Appendix G. It consists of equations [005], [006], [009], [010], [013], [015], [042] and [057], together with [067], [069] and [069],

$$PWINDEX = \sqrt{\frac{\sum_z PMT_z IMT_z^O}{\sum_z PMT_z^O IMT_z^O} \frac{\sum_z PMT_z IMT_z}{\sum_z PMT_z^O IMT_z}} \quad [066]$$

From the 2-region version of Model 3, consider the sub-model consisting of equations [005] and [009], together with

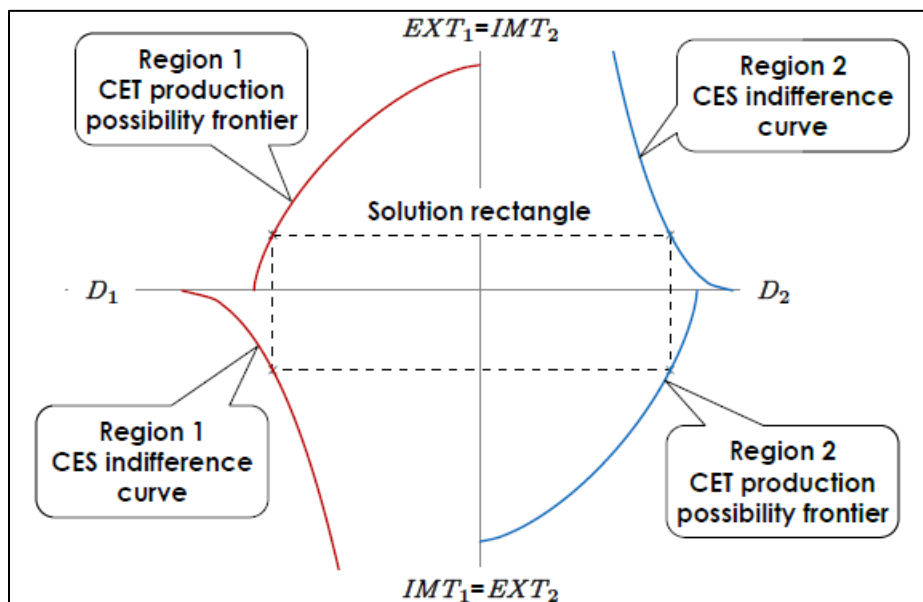
$$EXT_{zj} = IMT_z \quad [067]$$

Let us call this sub-model the Q-Model (Q for quantity). With two regions, there are 6 equations and 10 variables in the Q-model:  $XS_z$ ,  $D_z$ ,  $EXT_z$ ,  $Q_z$ ,  $IMT_z$ . In principle, it should be possible

to fix 4 of them, among which  $\overline{XS_z}$ .

Figure 2 summarizes the two-region Q-model. The position of the CET production frontiers in the north-west and south-east quadrants is determined by the fixed  $\overline{XS_z}$ . As long as the  $Q_z$  are not fixed, the CES indifference curves in the north-east and south-west quadrants are free to move inward or outward. Setting a pair of  $D_z$ ,  $EXT_z$ , or  $IMT_z$  specifies a point on each of the CET frontiers defined by  $\overline{XS_z}$ . Drawing perpendicular lines from these two points traces a rectangle. The solution corresponds to the values of  $Q_z$  that make the indifference curve pass through the north-east and south-west corners of the rectangle.

Figure 2 – The two-region Q-Model



It is quite obvious that if, *a contrario*, the  $Q_z$  are fixed rather than  $D_z$ ,  $EXT_z$ , or  $IMT_z$ , there is no guarantee that a solution rectangle exists. Therefore, given  $\overline{XS_z}$ ,  $Q_z$  cannot be fixed arbitrarily. And if a solution exists, it may well not be unique. Let us examine this more closely with the two-region Q-Model. Suppose the  $Q_z$  are fixed:  $Q_z = \overline{Q_z}$ . Then combining equations [005], [009] and [067] yields

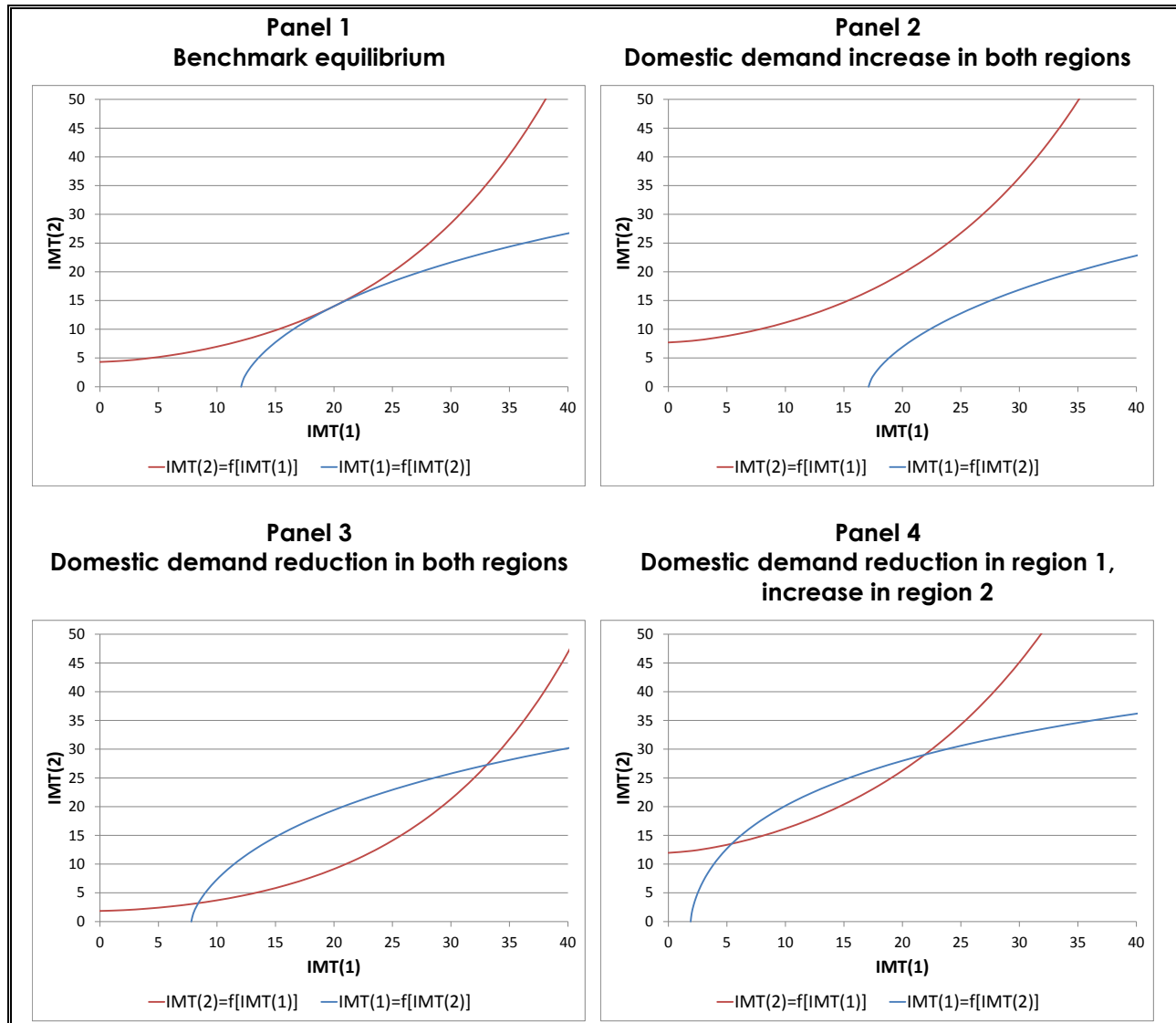
$$IMT_z = \left\{ \frac{1}{(1 - \alpha_z)} \left( \frac{\overline{Q_z}}{A_z} \right)^{-\rho_z} - \frac{\alpha_z}{(1 - \alpha_z)} \left[ \frac{1}{\beta_z} \left( \frac{\overline{XS_z}}{B_z} \right)^{\kappa_z} - \frac{(1 - \beta_z)}{\beta_z} IMT_{zj}^{\kappa_z} \right]^{-\frac{\rho_z}{\kappa_z}} \right\}^{-\frac{1}{\rho_z}} \quad [068]$$

(see Appendix H). Given  $\overline{XS_z}$  and  $\overline{Q_z}$ , the pair of equations [068] can be solved for  $IMT_z$ ; then  $EXT_z$  and  $D_z$  follow from [067] and either [009] or [005]. The pair of equations [068] ( $IMT_2$  as a function of  $IMT_1$ , and  $IMT_1$  as a function of  $IMT_2$ ) are plotted for different values of  $\overline{Q_z}$  in the four panels of Figure 3. The two functions are convex: as a function of the other region's imports, the imports of each region increases at a decreasing rate. The solution, of course, is given by the intersection of the two curves.

In the benchmark equilibrium, the curves are tangent at the solution (Panel 1). If the exogenous volume of domestic demand  $\overline{Q_z}$  is increased in both regions (Panel 2), the curves no longer

intersect: there is no solution. If, on the contrary,  $\overline{Q_z}$  is reduced in both regions (Panel 3), then the solution exists, but it is not unique. Finally, Panel 3 illustrates a situation where  $\overline{Q_z}$  is increased in one region, but reduced in the other; in that particular case, there is a solution, but it is not unique.

**Figure 3 – Existence of a solution in the two-region Q-Model**



This explains why, if one tries to close the model by fixing  $Q_z$ , then the CONOPT solver cannot find a solution and issues the diagnostic "Pivot too small", which "means that the set of constraints is linearly dependent in the current point and there is no unique search direction for Newton's method so CONOPT terminates" (CONOPT solver manual, p. 90).

What about closures that would fix  $D_z$ ,  $EXT_z$ , or  $IMT_z$ , in addition to  $\overline{XS_z}$ ? We have shown above that such a closure determines all quantity variables in the model, using equations [005], [009] and [067]. The issue is then whether there exists a set of prices that can make those quantities equilibrium quantities. Given the calibrated values of the model parameters, there is no guarantee that there is such a set of prices. Moreover, if such a set of prices exists, the price equation sub-system becomes degenerate.

To illustrate this, consider once again the two-region model described in Appendix G. After removing the Q-model equations, the rest of the model consists of 14 equations; given the Q-model solution, there are 15 remaining variables, one of which is to be fixed as the numéraire. Separate the 14 equations into two sub-models. The core price model (the P-Model) consists of 3 pairs of equations: equations [006] and [010], together with

$$PXT_{zj}^* = PMT_z^* \quad [069]$$

and it comprises 6 variables:  $PL_z^*$ ,  $PMT_z^*$ , and  $PXT_z^*$ . The remaining equations, [042], [057], [013], [015] and [066], constitute the downstream part of the model: it is readily solved using the solution values of the volume variables in the Q-Model and of the prices in the P-Model. Note, however, that the P-Model is homogenous in prices, as it should be. Therefore, its solution can be defined only up to a factor of proportionality. This implies logically that the 6 equations cannot be independent (the P-Model cannot be of full rank). Indeed, following the development in Appendix I, the P-Model equations may be combined to yield

$$\frac{PMT_z^*}{PMT_{zj}^*} = \frac{1 - \alpha_z}{\alpha_z} \left( \frac{D_z}{IMT_z} \right)^{1/\sigma_z} \frac{\beta_z}{1 - \beta_z} \left( \frac{D_z}{EXT_z} \right)^{1/\tau_z} \quad [070]$$

or, more explicitly,

$$\frac{PMT_1^*}{PMT_2^*} = \frac{1 - \alpha_1}{\alpha_1} \left( \frac{D_1}{IMT_1} \right)^{1/\sigma_1} \frac{\beta_1}{1 - \beta_1} \left( \frac{D_1}{EXT_1} \right)^{1/\tau_1} \quad [070.1]$$

$$\frac{PMT_1^*}{PMT_2^*} = \frac{\alpha_2}{1 - \alpha_2} \left( \frac{IMT_2}{D_2} \right)^{1/\sigma_2} \frac{1 - \beta_2}{\beta_2} \left( \frac{EXT_2}{D_2} \right)^{1/\tau_2} \quad [070.2]$$

Consequently, the P-Model has a solution only if

$$\frac{1 - \alpha_1}{\alpha_1} \left( \frac{D_1}{IMT_1} \right)^{1/\sigma_1} \frac{\beta_1}{1 - \beta_1} \left( \frac{D_1}{EXT_1} \right)^{1/\tau_1} = \frac{\alpha_2}{1 - \alpha_2} \left( \frac{IMT_2}{D_2} \right)^{1/\sigma_2} \frac{1 - \beta_2}{\beta_2} \left( \frac{EXT_2}{D_2} \right)^{1/\tau_2} \quad [071]$$

And if a solution exists, then [070] consists of two identical equations and the system is singular,

leading to “Pivot too small”. Note that equation [070] is implied by the model, and it therefore remains in force under any closure. What leads to the predicament described here is the fact that all right-hand side variables in [070] are hypothesized to have been predetermined in the Q-Model, so that they are treated as constants.

To summarize, we have shown for the two-region version of Model 3 that, given  $XS_z = \overline{XS_z}$ ,

- it is not possible to close the model using  $Q_z = \overline{Q_z}$ ;
- nor is it possible to close the model by fixing  $D_z$ ,  $EXT_z$ , or  $IMT_z$ .

I have not been able to formally generalize the demonstration to the  $N$ -region version of Model 3. But experiments show that the solver's behavior in the three-region version is the same as in the two-region version. And our close examination of the two-region Model 3 yields a rather powerful intuition of why the model doesn't solve when one attempts to fix volume variables in addition to  $XS_z = \overline{XS_z}$  in the closure (this includes volume variables  $EX_{z,zj}$  and  $IM_{z,zj}$ , which are absent in the two-region version).

#### 4.2.3 Price variable closures?

Finally, it should be obvious that fixing more than one price (the numéraire) in Model 3 is not admissible, because it prevents relative prices from adjusting.

## 5. Reintroduction of the exchange rates: Model 2 revisited

We now return to Model 2, to reconsider the issue of closures (see Tables 4 and 5). We use an expanded version of Model 2, as we did with Model 3, adding  $CABX_z$  and  $PWINDEX$  and the corresponding equations [057] and [058].

### 5.1 CLOSURE

The expanded Model 2 has  $10N + 3N(N - 1)$  equations like Model 3, and  $13N + 1 + 3N(N - 1)$  variables (recall that introducing equation [057] re-introduces  $CAB_{zleon}$ ). There are  $3N + 1$  degrees of freedom left,  $N$  more than in Model 3. We know that any solution of Model 3 can be extended to Model 2 using equations [050]-[055] to convert prices and nominal values from the international currency to regional currencies. Quite obviously, this transformation imposes no restriction on the  $e_z$  other than being strictly positive. Consequently, any set of conditions which constitutes a proper closure of Model 3, together with an arbitrary set of exchange rates, is also

a proper closure of Model 2<sup>14</sup>. One closure that is frequently used by modelers, which we call the FE closure (for Fixed Exchange rates), is to set the  $e_z$  and the  $CABX_z$  ( $2N$  variables fixed in the closure), and to pick some price such as  $PWINDEX$  or the reference region's producer price  $P_{zr}$  as the numéraire. As a matter of fact, fixing all exchange rates to 1 makes Model 2 numerically identical to Model 3.

Another possibility, which we call the FP closure (for Fixed regional Price indexes), is to fix the  $CABX_z$  and some set of regional prices, such as  $P_z$  ( $2N$  variables fixed in the closure), and to take a reference region's exchange rate  $e_{zr}$  or some international price as the numéraire. Why is it admissible to exogenously set prices other than the numéraire in Model 2, while it was not in Model 3? Because all that matters are the ratios of regional prices to exchange rates (equations [044]-[049]). In other words, if exchange rates  $e_z$  are free to move, then fixing some set of regional prices such as  $P_z$  does not prevent relative prices from adjusting any more than fixing  $e_z$ . And that remains true even if  $e_{zr}$  is chosen as the numéraire. However, no regional price, such as  $PL_{zr}$  for example, may play the role of numéraire, because then relative prices in region  $zr$  are prevented from adjusting. So with the FP closure, if the numéraire is not an exchange rate, then it must be an international price like  $PWINDEX$  or even some  $PW_{zr, zrj}$  (in which case the numéraire commodity is a particular bilateral trade flow between a pair of reference regions,  $zr$  and  $zrj$ ,  $zr, zrj \in \{1, \dots, N\}$ ).

But why consider the FP closure? Because it has an interesting interpretation. The fixed regional prices ( $P_z$  or other) play the part of regional numéraires<sup>15</sup>, and the  $e_z$  can then be viewed as real exchange rates<sup>16</sup>. As a matter of fact, if all regional prices ( $P_z$ ,  $PC_z$ , and  $PL_z$ ) in the solution are divided by the corresponding exchange rate, the values obtained are exactly equal to the regional prices obtained with the FE closure when  $e_z = 1$ : the endogenous

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<sup>14</sup> Fixing an exchange rate at a value different from its benchmark value merely results in a proportionally equal change in all regional prices and nominal variables (e.g. doubling region  $z$ 's fixed exchange rate will result in regional prices and nominal variables double their benchmark value).

<sup>15</sup> The fixed prices must be regional prices. They cannot be international prices: in particular, the  $PW_{z, zj}$  cannot be fixed, because they relate to a pair of regions, and not to a single region, so that they cannot play the role of regional numéraires.

<sup>16</sup> On the concept of real exchange rates, see Luis A. V. Catão (2007) "Why Real Exchange Rates?", *Finance & Development*, International Monetary Fund  
<http://www.imf.org/external/pubs/ft/fandd/2007/09/pdf/basics.pdf>  
 This article is part of the magazine's "Back to Basics" feature:  
<http://www.imf.org/external/pubs/ft/fandd/basics/>

exchange rates are indeed the ratio of domestic prices to international prices. Another advantage of the FP closure is that, in complex models, it facilitates the analysis of price changes. With regional numéraires, it is easier to distinguish between movements of regional prices relative to each other, and movements of regional prices together relative to international prices; the latter are reflected in the exchange rates.

Also note that, just like exchange rates are arbitrary when the numéraire is a price as in the FE closure, regional numéraires are arbitrary when the global numéraire is an exchange rate or an international price as in the FP closure.

Another point of interest is that the FE and FP closures mimic a fixed and flexible exchange rate regime respectively. I use the verb “mimic”, because in the idealized frictionless world of most CGE models, the difference as we have seen is purely formal. In the real world, however, things are quite different.

## **5.2 CALIBRATION CONSISTENCY**

### **5.2.1 Parametrization of Model 2**

Model 2 has  $14N + 2N(N - 1)$  parameters, listed in Table 8.

**Table 8 – Model 2 parameters**

$A_z$	Scale parameter, Armington CES function between local production and imports
$\alpha_z$	Share parameter, Armington CES function between local production and imports
$\sigma_z$	Elasticity of substitution between local production and imports: $0 < \sigma_z < \infty$
$\rho_z = \frac{1 - \sigma_z}{\sigma_z}$	: $-1 < \rho_z < \infty$
$A_z^M$	Scale parameter, Armington CES function between imports from different regions
$\alpha_{zj,z}^M$	Share parameter, Armington CES function between imports from different regions
$\sigma_z^M$	Elasticity of substitution between imports from different regions: $0 < \sigma_z^M < \infty$
$\rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M}$	: $-1 < \rho_z^M < \infty$
$B_z$	Scale parameter, CET product aggregator
$\beta_z$	Share parameter, CET product aggregator
$\tau_z$	Elasticity of transformation: $0 < \tau_z < \infty$
$\kappa_z = \frac{\tau_z + 1}{\tau_z}$	: $1 < \kappa_z < \infty$
$B_z^X$	Scale parameter, CET exports aggregator
$\beta_{z,zj}^X$	Share parameter, CET exports aggregator
$\tau_z^X$	Elasticity of transformation: $0 < \tau_z^X < \infty$
$\kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}$	: $1 < \kappa_z^X < \infty$

There are  $4N$  pairs (=  $8N$ ) of free parameters:  $(\sigma_z, \rho_z)$ ,  $(\sigma_z^M, \rho_z^M)$ ,  $(\tau_z, \kappa_z)$ , and  $(\tau_z^X, \kappa_z^X)$ . That leaves  $6N + 2N(N - 1)$  parameters to be calibrated. The expanded version of Model 2 has  $13N + 1 + 3N(N - 1)$  variables whose benchmark values must be determined, and  $10N + 3N(N - 1)$  equations. The SAM contains  $N^2$  transaction flows. Table 9 summarizes the situation.

**Table 9 – Constraints and unknowns in the calibration process**

Parameters	$14 N + 2 N (N - 1)$
Minus: Free parameters	$8 N$
Parameters to be calibrated	$6 N + 2 N (N - 1)$
Plus: Variables	$13 N + 1 + 3 N (N - 1)$
<b>Number of unknowns</b>	$19 N + 1 + 5 N (N - 1)$
<i>Equal to</i>	$5 N^2 + 14 N + 1$
Number of equations	$10 N + 3 N (N - 1)$
<i>Equal to</i>	$3 N^2 + 7 N$
Walras' Law	$1$
SAM transactions flows	$N^2$
Number of constraints	$4 N^2 + 7 N + 1$
Number of degrees of freedom	$N^2 + 7 N$

There are  $N^2 + 7 N$  degrees of freedom, which we dispose of as follows.

- Three sets of prices are fixed exogenously:  $e_z^O$ ,  $PL_z^O$  and  $PW_{z,zj}^O$ .
- We apply a normalization rule to define the benchmark value of four volume variables:  $EXT_z^O$ ,  $IMT_z^O$ ,  $Q_z^O$  and  $XS_z^O$  (see below about price  $\times$  volume factoring).
- Parameters  $\alpha_{zj,z}^M$  and  $\beta_{z,zj}^X$  are defined for each  $z$  only up to a factor of proportionality, and we apply the  $2 N$  normalization rules:  $\sum_{zj} \beta_{z,zj}^X = 1$  and  $\sum_{zj} \alpha_{zj,z}^M = 1$  (see Lemelin *et al.* 2013, Appendices D4.3.2.2 and D4.3.3.2).

**Table 10 – Additional constraints in the calibration process**

Exogenous prices $e_z^O$ , $PL_z^O$ and $PW_{z,zj}^O$	$2 N + N (N - 1)$
<i>Equal to</i>	$N^2 + N$
Parameter normalization rules for $\alpha_{zj,z}^M$ and $\beta_{z,zj}^X$	$2 N$
Composite volume variable normalization rules for $EXT_z^O$ , $IMT_z^O$ , $Q_z^O$ and $XS_z^O$	$4 N$
Number of additional constraints	$N^2 + 7 N$

### 5.2.2 Price × volume factoring

One implication of the arbitrariness of the price × volume factoring mentioned in 1.3 relates to the calibration of CES and CET aggregates<sup>17</sup>. Consider, for instance, the CET aggregate of exports to other regions.

Assume that  $e_z$ ,  $PW_{z,zj}$  and  $EX_{z,zj}$  have already been assigned their benchmark values  $e_z^O$ ,  $PW_{z,zj}^O$  and  $EX_{z,zj}^O$ , and that free parameters  $\tau_z^X$  and  $\kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}$  are defined. We want to

assign benchmark values to  $PXT_z$  and  $EXT_z$ , and to calibrate parameters  $B_z^X$  and  $\beta_{z,zj}^X$ : we have  $3N + N(N - 1)$  unknowns for  $2N + N(N - 1)$  equations, namely equations [007], [008] and [014]. That leaves  $N$  degrees of freedom, which correspond to the arbitrariness of the price × volume factoring. Choosing a normalization rule eliminates the extra degrees of freedom. For instance, we can pose

$$EXT_z^O = \sum_{zj} EX_{z,zj}^O \quad [072]$$

This is not restrictive. Indeed, observe that, if  $EXT_z^O$ ,  $PXT_z^O$  and  $B_z^X$  satisfy equations [007], [008] and [014], then so do  $EXT_z^{O2} = \lambda EXT_z^O$ ,  $PXT_z^{O2} = PXT_z^O / \lambda$  and  $B_z^{X2} = \lambda B_z^X$ , for any  $\lambda > 0$ .

Also note that the normalization rule [072] does not imply that exports to different regions are perfect substitutes. In simulations, the solution values will in general *not* satisfy equation [072].

Conversely, it would not be restrictive, rather than using [072], to fix

$$PXT_z^O = 1 \quad [073]$$

provided  $EXT_z^O$  is correctly calibrated according to [014], and the  $B_z^X$  are calibrated correspondingly (this is illustrated in the first calibration consistency test outlined in 6.2 below).

### 5.2.3 Calibration consistency and the calibration sequence

So model parametrization requires that additional constraints, such as [072] or [073], be imposed to solve the calibration problem. Following our definition of calibration consistency in 1.3 above, a model is said to be calibrated consistently if the relative variation of a variable between any simulation solution and its benchmark is the same, no matter which set of additional non-

---

<sup>17</sup> The reader is referred to Appendices D4.3.2 and D4.3.3 in Lemelin *et al.* (2013) for a mathematical exposition of the calibration of CES and CET function parameters.

restrictive constraints is imposed to solve the calibration problem. For this requirement to be fulfilled, the additional calibration constraints must be indeed neutral. But also, the calibration sequence must be constructed carefully to ensure that calibration is consistent with whichever particular set of additional constraints is applied.

This is illustrated in Table 11, which presents the calibration sequence of benchmark variables and parameters relating to exports.

**Table 11 – Calibration sequence of export related benchmark variables and parameters**

$e_z^O = 1$	[074]
$PI_z^O = 1$	[075]
$PW_{z,zj}^O = 1$	[076]
$D_z^O = \frac{\overline{PI_z^O D_z^O}}{PI_z^O} = \frac{\text{SAM transactions flow}}{PI_z^O}$	[077]
$EX_{z,zj}^O = \frac{\overline{PW_{z,zj}^O EX_{z,zj}^O}}{PW_{z,zj}^O} = \frac{\text{SAM transactions flow}}{PW_{z,zj}^O}$	[078]
$EXT_z^O = \sum_{zj} EX_{z,zj}^O$	[072]
$PXT_z^O = \frac{e_z^O \sum_{zj} PW_{z,zj}^O EX_{z,zj}^O}{EXT_z^O}$	[079]
$XS_z^O = D_z^O + EXT_z^O$	[080]
$P_z^O = \frac{PI_z^O D_z^O + PXT_z^O EXT_z^O}{XS_z^O}$	[081]
$\beta_z = \frac{PI_z^O D_z^O (1-\kappa_z)}{\left( PI_z^O D_z^O (1-\kappa_z) + PXT_z^O EXT_z^O (1-\kappa_z) \right)}$	[082]
$B_z = \frac{XS_z^O}{\left[ \beta_z (D_z^O)^{\kappa_z} + (1 - \beta_z) (EXT_z^O)^{\kappa_z} \right]^{\frac{1}{\kappa_z}}}$	[083]
$\beta_{z,zj}^X = \frac{PW_{z,zj}^O (EX_{z,zj}^O)^{1-\tau_z^X}}{\sum_{zjj} PW_{z,zjj}^O (EX_{z,zjj}^O)^{1-\tau_z^X}}$	[084]
$B_z^X = \frac{EXT_z^O}{\left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj}^O)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}}}$	[085]

Table 1 provides an example of the kind of pitfalls which can compromise calibration consistency. In view of [074], [076] and [072], it might be tempting to take a shortcut and replace [079] with  $PXT_z^O = 1$ ; that, however, would compromise calibration consistency, because the values assigned  $e_z^O$  and  $PW_{z,zj}^O$  are arbitrary, and not necessarily 1.

Note in particular that adopting a different rule than [072] would change the benchmark value of  $XS_z$  in [080], then of  $P_z$  in [081], and it would modify the calibrated value of  $B_z$ . So it is critical that these assignments be applied in the proper order. This is easy to verify here, but with a complex CGE model, the calibration procedure will be much more difficult to examine analytically.

Because of the convenience of calibrating aggregates as in [072], the calibration procedure of Model 2 applies the same rule to  $EXT_z^O$ ,  $IMT_z^O$ ,  $Q_z^O$  and  $XS_z^O$  (in that order). The calibration sequence of benchmark variables and parameters relating to imports is similar to that of exports. For a more complete presentation of the calibration procedure, the reader is referred to the GAMS code, which is amply documented with comments in the code itself. A more in-depth presentation of the parametrization strategy, as it is applied to PEP-w-1, can be found in Appendix D of Lemelin *et al.* (2013).

#### **5.2.4 Testing calibration consistency**

In a model as simple as Model 2, it is feasible to be reasonably certain of the consistency of the calibration procedure by diligent inspection of the code. In a complex model, even diligent inspection may overlook logical flaws, and testing offers an additional guarantee of consistency. To that end, the GAMS implementation of Model 2 contains options to test for calibration consistency.

## **6. GAMS implementation**

### **6.1 BRIEF DESCRIPTION OF GAMS PROGRAMS**

The GAMS programs can be downloaded from <http://www.pep-net.org/training-material-1>. They are found under "Other > CGE model closures in a skeleton world model".

Model 2 has been implemented in GAMS for illustration purposes. The standard version of the model is implemented in program *SkeletonWorld\_2014\_Model2A.gms*. The main program calls several sub-programs:

- *Calib\_check\_S2\_2014.gms* may be called at the end of calibration. It computes the difference between the left- and right-hand side of every equation when the variable arguments are replaced by their (calibrated) benchmark values.
- *Closures\_S2\_2014.gms* contains various closure options, including different choices of the numéraire and of its value. The user chooses by activating/disactivating the `$ontext/$offtext` switches in the program.
- *RESULTS\_BAU\_S2\_2014.gms* stores the BAU solution values.
- *Benchmk\_chk\_S2\_2014.gms* may be called once the BAU solution has been computed. It computes the difference between each variable's solution value and its benchmark (calibrated) value.
- *RESULTS\_SIM\_S2\_2014.gms* stores the SIM solution values and produces the GDX output file and its xls companion.
- *RATIOS\_S2\_2014.gms* computes the ratio of SIM solution values to BAU (benchmark) values for the purpose of checking for calibration consistency.

*Calib\_check\_S2\_2014.gms* and *Benchmk\_chk\_S2\_2014.gms* are not directly related to the issues discussed in this paper. They are two diagnostic tools frequently used by the author in developing models.

The program *SkeletonWorld\_2014\_Model2A.gms* produces two result files. The first is a “classic” Excel results file created from the standard GDX output file, using GDX2XLS, with one sheet per variable. The second is in tabular form, made with the GDXRW facility, reading parameters from a text file which is created within the GAMS model program.

There is a second program, *SkeletonWorld\_2014\_Model2B.gms*, which is basically the same, but offers in addition various possibilities for modifying the calibration and/or the closure rules, and making comparisons with the standard version of the model. Comparisons are made using two sub-programs:

- *Compare\_SOLUTIONS\_S2.gms* reads the standard model solution GDX file produced by *SkeletonWorld\_2014\_Model2A.gms*, and computes the ratios of Model2B/Model2A solution values, for the purpose of checking for model homogeneity when the choice of the numéraire, or its value, or both are modified. The output consists of files *Sol\_ratios.gdx* and *Sol\_ratios.xls*.
- *Compare\_RATIOS\_S2.gms* reads the GDX file of SIM/BAU ratios produced by *SkeletonWorld\_2014\_Model2A.gms*, and computes the Model2B–Model2A ratio differences, for the purpose of checking for calibration consistency. The output consists of files *Ratio\_diff.gdx* and *Ratio\_diff.xls*.

*SkeletonWorld\_2014\_Model2B.gms* is therefore a tool for testing model homogeneity and calibration consistency.

## 6.2 EXAMPLES OF TESTS

All the tests described here have been made against the standard version of the model with the FP closure and *PWINDEX* as the numéraire price:

```
XS.FX(z)      = XSO(z);
P.FX(z)       = PO(z);
CABX.FX(z)    = CABXO(z);
PWINDEX.FX    = PWINDEXO;
```

In Model2B, the closure is defined in *Closures\_S2B\_2014.gms*, as described for each test.

### 6.2.1 Test of FP and FE closures

The FE closure is implemented in Model2B:

```
XS.FX(z)      = XSO(z);
e.FX(z)       = eO(z);
CABX.FX(z)    = CABXO(z);
PWINDEX.FX    = PWINDEXO;
```

In *Compare\_SOLUTIONS\_S2.gms*, set the Lambda parameter to 1 and the program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol\_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical.

### 6.2.2 Homogeneity test 1

If a model is truly homogenous, the solution values of real (volume) variables and all price and nominal value ratios are supposed to be

- independent of which commodity is taken as the numéraire;
- independent of which region is taken as the reference region when the numéraire is a regional commodity (a particular case of the preceding);
- independent of the particular value given the price of the numéraire, whatever commodity plays that role.

In Model2B, implement the FP closure with  $PW_{zr, zrj}$  as the numéraire:

```
zr(z) = no;
zr('Reg1') = yes;
```

```

zrj(z) = no;
zrj('Reg2') = yes;

XS.FX(z)      = XSO(z);
P.FX(z)       = PO(z);
CABX.FX(z)    = CABXO(z);
PW.FX(zr,zrj) = PWO(zr,zrj);

```

In *Compare\_SOLUTIONS\_S2.gms*, set the  $\lambda$  parameter using<sup>18</sup>

```

Loop{(zr,zrj),
  Lambda(scen) = BvalPW(zr,zrj,scen)/valPW(zr,zrj,scen);
};

```

Then the program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol\_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical. As for exchange rates, their ratio is equal to  $1/\lambda$ . Now, let  $PL_z^A$  be the value of  $PL_z$  in the Model2A solution, and  $PL_z^B$  its value in the Model2B solution. We observe in *Sol\_ratios.xls* that

$$\frac{(PL_z^B/e_z^B)}{\lambda (PL_z^A/e_z^A)} = 1 \quad [086]$$

$$\frac{e_z^B}{e_z^A} = \frac{1}{\lambda} \quad [087]$$

Consequently,

$$\frac{PL_z^B}{PL_z^A} = 1 \quad [088]$$

which is as it should be under the FP closure. The reason is that the regional numéraires (here  $P_z$ ) are fixed, so that, if the models are identical, going from *PWINDEX* to  $PW_{zr,zrj}$  as the numéraire leaves regional prices unchanged.

Homogeneity test 1 can be performed with the same results if the numéraire is given any arbitrary positive value. For example,

```

PW.FX(zr,zrj)      = 1.7*PWO(zr,zrj);

```

However, with multiples outside the [0.45,1.7] range, the model needs to be initialized accordingly for GAMS to be able to solve it:

---

<sup>18</sup> The loop is necessary because *zr* and *zrj* are sets (albeit singletons) in GAMS.

```

PW.FX(zr, zrj)      =  Lambda0*PWO(zr, zrj);

e.L(z)              =  eO(z)/Lambda0;
PW.L(z, zj)         =  Lambda0*PWO(z, zj);
PWINDEX.L          =  Lambda0*PWINDEXO;

```

(the re-initialization code appears in *Closures\_S2\_2014.gms*).

### 6.2.3 Homogeneity test 2

In Model2B, implement the FP closure with  $e_{zr}$  as the numéraire:

```

zr(z) = no;
zr('Reg1') = yes;

XS.FX(z)      =  XSO(z);
P.FX(z)       =  PO(z);
CABX.FX(z)    =  CABXO(z);
e.FX(zr)      =  eO(zr);

```

In *Compare\_SOLUTIONS\_S2.gms*, set the  $\lambda$  parameter using<sup>19</sup>

```

Loop{zr,
  Lambda(scen) = vale(zr, scen)/Bvale(zr, scen);
};

```

which is equivalent to

$$\lambda = \frac{(1/e_{zr}^B)}{(1/e_{zr}^A)} \quad [089]$$

This reflects the fact that the numéraire is an international price: it is actually  $1/e_{zr}$ , the price of region  $zr$ 's currency in terms of the international currency, rather than  $e_{zr}$ , the price of the international currency in terms of region  $zr$ 's currency.

The program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol\_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical.

Homogeneity test 2 can be performed with the same results if the numéraire is given any arbitrary positive value, but with large or small multiples, the model needs to be initialized accordingly for GAMS to be able to solve it:

---

<sup>19</sup> The loop is necessary because  $zr$  and  $zrj$  are sets (albeit singletons) in GAMS.

```
e.FX(zr)      = Lambda0*eO(zr);
e.L(z)        = Lambda0*eO(z);
PW.L(z,zj)    = PWO(z,zj)/Lambda0;
PWINDEX.L     = PWINDEXO/Lambda0;
```

### 6.2.4 Homogeneity test 3

In Model2B, implement the FP closure with *PWINDEX* as the numéraire and change  $P_z$  for  $PL_z$  as the regional numéraire:

```
XS.FX(z)      = XSO(z);
PL.FX(z)      = PLO(z);
CABX.FX(z)    = CABXO(z);
PWINDEX.FX    = Lambda0*PWINDEXO;
```

and add the corresponding re-initialization:

```
e.L(z)        = eO(z)/Lambda0;
PW.L(z,zj)    = Lambda0*PWO(z,zj);
```

In *Compare\_SOLUTIONS\_S2.gms*, set the  $\lambda$  parameter using

```
Lambda(scen)  = BvalPWINDEX(scen)/valPWINDEX(scen);
```

The program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol\_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical.

### 6.2.5 Homogeneity test 4

In Model2B, implement the FP closure with *PWINDEX* as the numéraire and change  $P_z$  for  $PL_z$  as the regional numéraire:

```
XS.FX(z)      = XSO(z);
PL.FX(z)      = Lambda0*PLO(z);
CABX.FX(z)    = CABXO(z);
PWINDEX.FX    = PWINDEXO;
```

and add the corresponding re-initialization:

```
e.L(z)        = Lambda0*eO(z);
P.L(z)        = Lambda0*PO(z);
PC.L(z)       = Lambda0*PCO(z);
PMT.L(z)      = Lambda0*PMTO(z);
```

```
PXT.L(z)      = Lambda0*PXTO(z);
CAB.L(z)      = Lambda0*CABO(z);
```

In *Compare\_SOLUTIONS\_S2.gms*, set the  $\lambda$  parameter using

```
Lambda(scen)  = BvalPWINDEX(scen)/valPWINDEX(scen);
```

The program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol\_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical.

### 6.2.6 Homogeneity test 5 with FE closure

The modified FE closure is implemented in Model2B with  $P_{zr}$  rather than *PWINDEX* as numéraire:

```
XS.FX(z)      = XSO(z);
e.FX(z)       = eO(z);
CABX.FX(z)    = CABXO(z);
P.FX(zr)      = Lambda0*PO(zr);
```

and add the corresponding re-initialization:

```
PL.L(z)       = Lambda0*PLO(z);
PC.L(z)       = Lambda0*PCO(z);
PMT.L(z)      = Lambda0*PMTO(z);
PXT.L(z)      = Lambda0*PXTO(z);
PW.L(z,zj)    = Lambda0*PWO(z,zj);
PWINDEX.L     = Lambda0*PWINDEXO;
CAB.L(z)      = Lambda0*CABO(z);
```

In *Compare\_SOLUTIONS\_S2.gms*, set the  $\lambda$  parameter using

```
Loop{zr,
  Lambda(scen) = [BvalP(zr,scen)/Bvale(zr,scen)]
                / [valP(zr,scen)/vale(zr,scen)];
};
```

The program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol\_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical.

### 6.2.7 Calibration consistency test of price × volume factoring

Implement the standard FP closure in Model2B with *PWINDEX* as the numéraire. Then, to perform the test, go to section 4.1 of the *SkeletonWorld\_2014\_Model2B.gms* main program, look for the line

```
*$GoTo Alternate_calib
```

and remove the asterisk at the beginning. In *Compare\_SOLUTIONS\_S2.gms*, set the  $\lambda$  parameter using

```
Lambda(scen) = 1;
```

The program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol\_ratios.xls* that the ratios of the prices of aggregates ( $PXT_z$ ,  $PMT_z$ ,  $PC_z$  and  $P_z$ ) are equal to the Model2B/Model2A ratios of their benchmark values, while the ratios of the corresponding volumes are the inverse, including in the SIM results. In *Ratio\_diff.xls*, observe that the SIM/BAU ratios are equal in both models, which satisfies the criterion of calibration consistency as stated in equations [001] and [002] of section 1.3:

### 6.2.8 Calibration consistency test of arbitrary prices

Implement the standard FP closure in Model2B with *PWINDEX* as the numéraire.

In section 3.4 of the *SkeletonWorld\_2014\_Model2B.gms* program, replace the following statements

```
eO(z)      = 1;
PLO(z)     = 1;
PWO(z, zj) = 1;
```

with different assignments. For example:

```
eO(z)      = .5;
PLO(z)     = 0.8;
PWO(z, zj) = 1.5;
PLO('Reg3') = 1.6;
PWO(zr, zrj) = 2;
```

In *Sol\_ratios.xls*, the SIM ratios and the BAU ratios are equal, and the passionate reader could trace the sources of divergence from 1. More interesting is *Ratio\_diff.xls*, where it can be verified

that the SIM/BAU ratios are equal in both models, which satisfies the criterion of calibration consistency as stated in section 1.3.

### **6.2.9 Calibration consistency test of arbitrary prices and exchange rates**

Implement the standard FP closure in Model2B with *PWINDEX* as the numéraire. In the *SkeletonWorld\_2014\_Model2B.gms* program, go to the section labeled “4.1\_supplement Re-calibration with arbitrary unequal exchange rates”, and activate the procedure by cancelling the \$ontext/\$offtext switches. The same result obtains as in the preceding test.

## **Conclusion**

This paper was born out of the author's personal experience in developing a worldwide CGE model. At times, our team was confronted with pairs of model solutions that should be equivalent in theory, but didn't look like they were. In the course of (successfully) resolving such paradoxes, we feel we have gained a deeper understanding of the issues discussed here, and of CGE modelling in general. We want to share our insights, because the issues which we explore in this paper are seldom discussed explicitly in the descriptions of global multinational models. For that purpose, we develop a highly simplified skeleton model derived from the PEP-w-1 worldwide CGE model (Lemelin *et al.*, 2013), but which represents the essential structure of several world trade models.

The issues discussed may be grouped into three sets. First, in order to determine how many variables must be exogenously fixed in the closure, we delve into the identification of redundant equations. Beginning with a set of theoretical equations, the model is pared down step by step by eliminating one set of redundant equations after another, making sure in the process to avoid circular reasoning. The final step is to highlight Walras' Law. In all CGE models, one equilibrium relation is routinely discarded, and a numéraire is defined, invoking Walras' Law. But seldom is Walras' Law explicitly deducted from model equations as it is done above.

Secondly, we consider model closures and homogeneity with respect to prices, the role of exchange rates, and the way they are dealt with in closing the model. To begin with, we demonstrate that exchange rates can be dispensed with in a non-monetary CGE model (Model 3). Next, we show that fixed nominal current account balances, a widely used closure rule, is in principle incorrect because it compromises model homogeneity (although it may be correct under certain conditions), and we formulate the theoretically correct fixed current account closure. Then, we broaden our view to see which variables other than  $CABX_z$  could be made exogenous to close the model: we find that in the highly simplified skeleton model without

exchange rates (Model 3), once the neoclassical full-employment hypothesis is applied ( $XS_z = \overline{XS_z}$ ), there is really no other closure possible, except for variations on the numéraire. Finally, if a model does retain exchange rate variables (Model 2), then we prove that they can be given arbitrary values. We go on to show that the supplementary degrees of freedom that are created by introducing exchange rates in the model allow for defining what we have labeled the FP closure (Fixed regional Price indexes), in which there is a fixed local numéraire for each region, and the endogenous exchange rates can be interpreted as *real* exchange rates. And we establish that the model under the FP closure is equivalent to the model under the FE closure. Put otherwise, the two model closures are different in their implementation, but are mathematically and economically equivalent.

The third set of questions examined relates to calibration consistency. Even after so-called “free” parameters (elasticities and the like) have been specified, the calibration problem usually remains under-determined. Consequently, additional constraints (arbitrary exogenous values and normalization rules) are imposed. Such additional constraints, given that they are based neither on observation (SAM flows), nor on theory (model equations), should be neutral, or non-restrictive. Moreover, whichever set of neutral additional constraints are used to make the calibration problem determinate, the calibration procedure should yield parametrized models that are equivalent. This is what calibration consistency is about.

This paper is completed by a set of GAMS programs which implement the model, and allow for experimentation with alternate calibration procedures and model closures, and test for model homogeneity and calibration consistency. These exercises may feel like mental gymnastics and puzzles over byzantine matters. And perhaps they are. But I do hope that they will help the reader as it has helped me, to gain a better understanding... and have fun along the way!

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## Appendices

**NOTE regarding the numbering of equations:** In the appendices, new equation numbers begin with a letter designating the appendix in which it appears. Equations that are cited from the main text bear the same number as in the main text.

### APPENDIX A: DETAILED STATEMENT OF MODEL 1

In this appendix, we give a detailed statement of Model 1.

#### Production

With fixed production factors and full employment, output is fixed in each region  $z$ .

$$XS_z = \overline{XS_z} \quad [056]$$

where

$XS_z$  Domestic production in region  $z$

Equation [056], however, is not part of the model; rather, it is a closure equation.

#### Income and savings

Regional income is equal to the value of production.

$$Y_z = P_z XS_z \quad [A001]$$

where

$P_z$  Producer price in region  $z$

$Y_z$  Income in region  $z$

There is no distinction between consumption and investment. It follows that savings, the difference between income and consumption, are equal to the current account balance (CAB). So the regional agent's budget constraint is

$$CAB_z = Y_z - PC_z Q_z \quad [A002]$$

where

$CAB_z$  Current account balance of region  $z$

$PC_z$  Price of the composite good in region  $z$

$Q_z$  Domestic demand for the composite good in region  $z$

To make the model more compact, we substitute [A001] into [A002], which becomes

$$CAB_z = P_z XS_z - PC_z Q_z \quad [003]$$

We can eliminate equation [A001] and variable  $Y_z$  from the model.

Not only is the current account balance equal to savings, but it is by definition equal to the difference between the aggregate value of exports and the aggregate value of imports.

$$CAB_z = PXT_z EXT_z - PMT_z IMT_z \quad [004]$$

where

- $IMT_z$  Total imports of region z
- $EXT_z$  Total exports of region z
- $PMT_z$  Price of composite imports in region z
- $PXT_z$  Price of composite exports of region z

### Trade

Production is allocated between sales on the domestic market and exports so as to maximize its value subject to a CET transformation function.

$$XS_z = B_z \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} \quad \text{where } \kappa_z = \frac{\tau_z + 1}{\tau_z}, \text{ with } 0 < \tau_z < \infty \quad [005]$$

$$\frac{EXT_z}{D_z} = \left( \frac{\beta_z PXT_z}{1 - \beta_z PL_z} \right)^{\tau_z} \quad [006]$$

where

- $D_z$  Domestic demand for the locally produced good in region z
- $PL_z$  Market price of local product in region z

and

- $B_z$  Scale parameter, CET product aggregator
- $\beta_z$  Share parameter, CET product aggregator
- $\tau_z$  Elasticity of transformation:  $0 < \tau_z < \infty$

$$\kappa_z = \frac{\tau_z + 1}{\tau_z} : 1 < \kappa_z < \infty$$

Total exports are allocated among destination regions so as to maximize their value subject to a CET transformation function.

$$EXT_z = B_z^X \left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} \quad \text{where } \kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}, \text{ with } 0 < \tau_z^X < \infty \quad [007]$$

$$EX_{z,zj} = \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \left[ \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X PXT_z} \right]^{\tau_z^X} \quad [008]$$

where

- $B_z^X$  Scale parameter, CET exports aggregator  
 $\beta_{z,zj}^X$  Share parameter, CET exports aggregator  
 $\tau_z^X$  Elasticity of transformation:  $0 < \tau_z^X < \infty$

$$\kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X} : 1 < \kappa_z^X < \infty$$

- $e_z$  Exchange rate (price of the international currency in terms of region z's currency)  
 $PW_{z,zj}$  World price of exports from region z to region zj  
 $EX_{z,zj}$  Exports by region z to region zj

Under the Armington hypothesis, domestic demand is distributed between the domestically produced good and imports so as to maximize the quantity acquired, subject to a CES aggregator function.

$$Q_z = A_z \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} \text{ where } \rho_z = \frac{1 - \sigma_z}{\sigma_z}, \text{ with } 0 < \sigma_z < \infty \quad [009]$$

$$\frac{IMT_z}{D_z} = \left( \frac{1 - \alpha_z}{\alpha_z} \frac{PL_z}{PMT_z} \right)^{\sigma_z} \quad [010]$$

where

- $A_z$  Scale parameter, Armington CES function between local production and imports  
 $\alpha_z$  Share parameter, Armington CES function between local production and imports  
 $\sigma_z$  Elasticity of substitution between local production and imports:  $0 < \sigma_z < \infty$

$$\rho_z = \frac{1 - \sigma_z}{\sigma_z} : -1 < \rho_z < \infty$$

Under the Armington hypothesis, imports are distributed among exporting regions so as to maximize the quantity acquired, subject to a CES aggregator function.

$$IMT_z = A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M} \right]^{-\frac{1}{\rho_z^M}} \text{ where } \rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M}, \text{ with } 0 < \sigma_z^M < \infty \quad [011]$$

$$IM_{zj,z} = \frac{IMT_z}{(A_z^M)^{1 - \sigma_z^M}} \left[ \frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right]^{\sigma_z^M} \quad [012]$$

where

- $A_z^M$  Scale parameter, Armington CES function between imports from different regions

$\alpha_{zj,z}^M$  Share parameter, Armington CES function between imports from different regions

$\sigma_z^M$  Elasticity of substitution between imports from different regions:  $0 < \sigma_z^M < \infty$

$$\rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M} : -1 < \rho_z^M < \infty$$

### Prices

The value of production is equal to the sum of the value of sales on the domestic market and exports.

$$P_z X S_z = P L_z D_z + P X T_z E X T_z \quad [013]$$

The total value of exports is equal to the sum of values of exports to all regions

$$P X T_z E X T_z = e_z \sum_{zj} P W_{z,zj} E X_{z,zj} \quad [014]$$

Total expenditures are equal to the sum of purchases on the domestic market and the value of imports.

$$P C_z Q_z = P L_z D_z + P M T_z I M T_z \quad [015]$$

The total value of imports is equal to the sum of values of imports from all regions

$$P M T_z I M T_z = e_z \sum_{zj} P W_{zj,z} I M_{zj,z} \quad [016]$$

### Equilibrium

Imports from region  $zj$  by region  $z$  must be equal to exports from region  $zj$  to region  $z$ .

$$E X_{zj,z} = I M_{zj,z} \quad [017]$$

The world sum of current account balances, expressed in the international currency, must be zero.

$$\sum_z \frac{C A B_z}{e_z} = 0 \quad [018]$$



## APPENDIX B: REDUNDANCY OF EQUATIONS [014] AND [016]

Given [007] and [008], equation [014] is redundant, and given [011] and [012], equation [016] is redundant. This is demonstrated in parallel in three steps.

**Step 1:** Substitute [008] into [007], and [012] into [011], and develop.

$$EXT_z = B_z^X \left[ \sum_{zj} \beta_{z,zj}^X \left( \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \left[ \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X PXT_z} \right]^{\tau_z^X} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}}$$

$$EXT_z = B_z^X \left[ \sum_{zj} \beta_{z,zj}^X \left( \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \left[ \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X PXT_z} \right]^{\tau_z^X} \right)^{\frac{\tau_z^X+1}{\tau_z^X}} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$EXT_z = B_z^X \left[ \left( \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \right)^{\frac{\tau_z^X+1}{\tau_z^X}} \sum_{zj} \beta_{z,zj}^X \left( \left[ \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X PXT_z} \right]^{\tau_z^X} \right)^{\frac{\tau_z^X+1}{\tau_z^X}} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$EXT_z = B_z^X \left( \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \right) \left[ \sum_{zj} \beta_{z,zj}^X \left( \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X PXT_z} \right)^{\tau_z^X+1} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$1 = \frac{1}{(B_z^X)^{\tau_z^X}} \left[ \sum_{zj} \beta_{z,zj}^X \left( \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X PXT_z} \right)^{\tau_z^X+1} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$IMT_z = A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M \left( \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} \left[ \frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right]^{\sigma_z^M} \right)^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} \quad [B001]$$

$$IMT_z = A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M \left( \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} \left[ \frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right]^{\sigma_z^M} \right)^{-\frac{1-\sigma_z^M}{\sigma_z^M}} \right]^{\frac{-\sigma_z^M}{1-\sigma_z^M}} \quad [B002]$$

$$IMT_z = A_z^M \left[ \left( \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} \right)^{\frac{1-\sigma_z^M}{\sigma_z^M}} \sum_{zj} \alpha_{zj,z}^M \left( \left[ \frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right]^{\sigma_z^M} \right)^{\frac{1-\sigma_z^M}{\sigma_z^M}} \right]^{\frac{-\sigma_z^M}{1-\sigma_z^M}} \quad [B003]$$

$$IMT_z = A_z^M \left( \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} \right) \left[ \sum_{zj} \alpha_{zj,z}^M \left( \frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right)^{\sigma_z^M-1} \right]^{\frac{-\sigma_z^M}{1-\sigma_z^M}} \quad [B004]$$

$$1 = \frac{1}{(A_z^M)^{-\sigma_z^M}} \left[ \sum_{zj} \alpha_{zj,z}^M \left( \frac{e_z PW_{zj,z}}{\alpha_{zj,z}^M PMT_z} \right)^{1-\sigma_z^M} \right]^{\frac{-\sigma_z^M}{1-\sigma_z^M}} \quad [B005]$$

$$1 = \frac{1}{(B_z^X)^{\tau_z^X}} \left[ \left( \frac{1}{PXT_z} \right)^{\tau_z^X+1} \sum_{zj} \beta_{z,zj}^X \left( \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X+1} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$1 = \frac{1}{(B_z^X)^{\tau_z^X}} \left( \frac{1}{PXT_z} \right)^{\tau_z^X} \left[ \sum_{zj} \beta_{z,zj}^X \left( \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X+1} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$(PXT_z)^{\tau_z^X} = \frac{1}{(B_z^X)^{\tau_z^X}} \left[ \sum_{zj} \beta_{z,zj}^X \left( \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X+1} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$PXT_z = \frac{1}{B_z^X} \left[ \sum_{zj} \beta_{z,zj}^X \left( \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X+1} \right]^{\frac{1}{\tau_z^X+1}}$$

$$PXT_z = \frac{1}{B_z^X} \left[ \sum_{zj} \left( \frac{1}{\beta_{z,zj}^X} \right)^{\tau_z^X} (e_z PW_{z,zj})^{\tau_z^X+1} \right]^{\frac{1}{\tau_z^X+1}}$$

$$1 = \frac{1}{(A_z^M)^{-\sigma_z^M}} \left[ \left( \frac{1}{PMT_z} \right)^{1-\sigma_z^M} \sum_{zj} \alpha_{zj,z}^M \left( \frac{e_z PW_{zj,z}}{\alpha_{zj,z}^M} \right)^{1-\sigma_z^M} \right]^{\frac{-\sigma_z^M}{1-\sigma_z^M}} \quad [B006]$$

$$1 = \frac{1}{(A_z^M)^{-\sigma_z^M}} \left( \frac{1}{PMT_z} \right)^{\sigma_z^M} \left[ \sum_{zj} \alpha_{zj,z}^M \left( \frac{e_z PW_{zj,z}}{\alpha_{zj,z}^M} \right)^{1-\sigma_z^M} \right]^{\frac{\sigma_z^M}{1-\sigma_z^M}} \quad [B007]$$

$$(PMT_z)^{-\sigma_z^M} = \frac{1}{(A_z^M)^{-\sigma_z^M}} \left[ \sum_{zj} \alpha_{zj,z}^M \left( \frac{e_z PW_{zj,z}}{\alpha_{zj,z}^M} \right)^{1-\sigma_z^M} \right]^{\frac{-\sigma_z^M}{1-\sigma_z^M}} \quad [B008]$$

$$PMT_z = A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M \left( \frac{e_z PW_{zj,z}}{\alpha_{zj,z}^M} \right)^{1-\sigma_z^M} \right]^{\frac{1}{1-\sigma_z^M}} \quad [B009]$$

$$PMT_z = A_z^M \left[ \sum_{zj} (\alpha_{zj,z}^M)^{\sigma_z^M} (e_z PW_{zj,z})^{1-\sigma_z^M} \right]^{\frac{1}{1-\sigma_z^M}} \quad [B010]$$

**Step 2:** Multiply both sides of [008] by  $e_z PW_{z,zj}$ , and both sides of [012] by  $e_z PW_{zj,z}$ , and sum over  $zj$ .

$$e_z \sum_{zj} PW_{z,zj} EX_{z,zj} = e_z \sum_{zj} PW_{z,zj} \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \left[ \frac{e_z PW_{z,zj}}{\beta_{z,zj}^X PXT_z} \right]^{\tau_z^X}$$

$$e_z \sum_{zj} PW_{z,zj} EX_{z,zj} = \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \left( \frac{1}{PXT_z} \right)^{\tau_z^X} \sum_{zj} (e_z PW_{z,zj})^{1+\tau_z^X} \left( \frac{1}{\beta_{z,zj}^X} \right)^{\tau_z^X}$$

$$\sum_{zj} (e_z PW_{z,zj})^{1+\tau_z^X} \left( \frac{1}{\beta_{z,zj}^X} \right)^{\tau_z^X} = e_z \sum_{zj} PW_{z,zj} EX_{z,zj} \left( \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \left( \frac{1}{PXT_z} \right)^{\tau_z^X} \right)^{-1}$$

$$e_z \sum_{zj} PW_{zj,z} IM_{zj,z} = \sum_{zj} e_z PW_{zj,z} \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} \left[ \frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right]^{\sigma_z^M} \quad [B011]$$

$$e_z \sum_{zj} PW_{zj,z} IM_{zj,z} = \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} (PMT_z)^{\sigma_z^M} \sum_{zj} (e_z PW_{zj,z})^{1-\sigma_z^M} (\alpha_{zj,z}^M)^{\sigma_z^M} \quad [B012]$$

$$\sum_{zj} (e_z PW_{zj,z})^{1-\sigma_z^M} (\alpha_{zj,z}^M)^{\sigma_z^M} = e_z \sum_{zj} PW_{zj,z} IM_{zj,z} \left( \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} (PMT_z)^{\sigma_z^M} \right)^{-1} \quad [B013]$$

**Step 3:** Substitute the right-hand side of [B013] into [B010]

$$PXT_z = \frac{1}{B_z^X} \left[ e_z \sum_{zj} PW_{z,zj} EX_{z,zj} \left( \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \left( \frac{1}{PXT_z} \right)^{\tau_z^X} \right)^{-1} \right]^{\frac{1}{\tau_z^X+1}}$$

$$PXT_z = \frac{1}{B_z^X} \left( \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \left( \frac{1}{PXT_z} \right)^{\tau_z^X} \right)^{\frac{1}{\tau_z^X+1}} \left[ e_z \sum_{zj} PW_{z,zj} EX_{z,zj} \right]^{\frac{1}{\tau_z^X+1}}$$

$$PXT_z = \frac{1}{B_z^X} \left( \frac{EXT_z}{(B_z^X)^{1+\tau_z^X}} \right)^{-\frac{1}{\tau_z^X+1}} \left( \frac{1}{PXT_z} \right)^{\frac{\tau_z^X}{\tau_z^X+1}} \left[ e_z \sum_{zj} PW_{z,zj} EX_{z,zj} \right]^{\frac{1}{\tau_z^X+1}}$$

$$(PXT_z EXT_z)^{\frac{1}{\tau_z^X+1}} = \left[ e_z \sum_{zj} PW_{z,zj} EX_{z,zj} \right]^{\frac{1}{\tau_z^X+1}}$$

$$PXT_z EXT_z = e_z \sum_{zj} PW_{z,zj} EX_{z,zj}$$

$$PMT_z = A_z^M \left[ e_z \sum_{zj} PW_{zj,z} IM_{zj,z} \left( \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} (PMT_z)^{\sigma_z^M} \right)^{-1} \right]^{\frac{1}{1-\sigma_z^M}} \quad [B014]$$

$$PMT_z = A_z^M \left( \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} (PMT_z)^{\sigma_z^M} \right)^{-\frac{1}{1-\sigma_z^M}} \left[ e_z \sum_{zj} PW_{zj,z} IM_{zj,z} \right]^{\frac{1}{1-\sigma_z^M}} \quad [B015]$$

$$PMT_z = A_z^M \left( \frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}} \right)^{-\frac{1}{1-\sigma_z^M}} \left( \frac{1}{PMT_z} \right)^{\frac{\sigma_z^M}{1-\sigma_z^M}} \left[ e_z \sum_{zj} PW_{zj,z} IM_{zj,z} \right]^{\frac{1}{1-\sigma_z^M}} \quad [B016]$$

$$(PMT_z IMT_z)^{\frac{1}{1-\sigma_z^M}} = \left[ e_z \sum_{zj} PW_{zj,z} IM_{zj,z} \right]^{\frac{1}{1-\sigma_z^M}} \quad [B017]$$

$$PMT_z IMT_z = e_z \sum_{zj} PW_{zj,z} IM_{zj,z} \quad [B018]$$

Equation [B018] is identical to [014] and [016], which are therefore redundant.



## APPENDIX C: EQUATION [025]

### C.1 Cost minimizing problem

The regional agent allocates demand between domestic production and imports by minimizing consumption expenditures

$$PC_z Q_z = PL_z D_z + PMT_z IMT_z \quad [015]$$

subject to

$$Q_z = A_z \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} \text{ where } \rho_z = \frac{1 - \sigma_z}{\sigma_z}, \text{ with } 0 < \sigma_z < \infty \quad [009]$$

### C.2 Lagrangian and first-order conditions

Form the Lagrangian

$$\mathcal{L}_z = PL_z D_z + PMT_z IMT_z - \mu_z \left\{ A_z \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} - Q_z \right\} \quad [C001]$$

The first order conditions are:

$$\frac{\partial \mathcal{L}_z}{\partial \mu_z} = -A_z \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} + Q_z = 0 \quad [C002]$$

$$\frac{\partial \mathcal{L}_z}{\partial D_z} = PL_z - \mu_z \frac{\partial}{\partial D_z} A_z \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} = 0 \quad [C003]$$

$$\frac{\partial \mathcal{L}_z}{\partial IMT_z} = PMT_z - \mu_z \frac{\partial}{\partial IMT_z} A_z \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} = 0 \quad [C004]$$

with

$$\begin{aligned} \frac{\partial}{\partial D_z} A_z \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} = \\ - \frac{A_z}{\rho_z} \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}-1} \frac{\partial}{\partial D_z} \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right] \end{aligned} \quad [C005]$$

$$\begin{aligned} \frac{\partial}{\partial D_z} A_z [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{-\frac{1}{\rho_z}} &= \\ &+ A_z [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{-\frac{\rho_z+1}{\rho_z}} \alpha_z D_z^{-\rho_z-1} \end{aligned} \quad [C006]$$

and

$$\begin{aligned} \frac{\partial}{\partial IMT_z} A_z [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{-\frac{1}{\rho_z}} &= \\ &- \frac{A_z}{\rho_z} [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{-\frac{1}{\rho_z}-1} \frac{\partial}{\partial IMT_z} [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}] \end{aligned} \quad [C007]$$

$$\begin{aligned} \frac{\partial}{\partial IMT_z} A_z [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{-\frac{1}{\rho_z}} &= \\ &+ A_z [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{-\frac{\rho_z+1}{\rho_z}} (1 - \alpha_z) IMT_z^{-\rho_z-1} \end{aligned} \quad [C008]$$

Given [009],

$$\frac{Q_z}{A_z} = [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{-\frac{1}{\rho_z}} \quad [C009]$$

$$\left( \frac{Q_z}{A_z} \right)^{-\rho_z} = [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}] \quad [C010]$$

Substitute [C010] into [C006] to obtain

$$\frac{\partial}{\partial D_z} A_z [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{-\frac{1}{\rho_z}} = A_z \left[ \left( \frac{Q_z}{A_z} \right)^{-\rho_z} \right]^{-\frac{\rho_z+1}{\rho_z}} \alpha_z D_z^{-\rho_z-1} \quad [C011]$$

$$\frac{\partial}{\partial D_z} A_z [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{-\frac{1}{\rho_z}} = A_z \left( \frac{Q_z}{A_z} \right)^{\rho_z+1} \alpha_z D_z^{-\rho_z-1} \quad [C012]$$

$$\frac{\partial}{\partial D_z} A_z [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{-\frac{1}{\rho_z}} = A_z^{-\rho_z} \alpha_z \left( \frac{Q_z}{D_z} \right)^{\rho_z+1} \quad [C013]$$

and similarly into [C008], yielding

$$\frac{\partial}{\partial IMT_z} A_z [\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z}]^{-\frac{1}{\rho_z}} = A_z^{-\rho_z} (1 - \alpha_z) \left( \frac{Q_z}{IMT_z} \right)^{\rho_z+1} \quad [C014]$$

First-order conditions [C003] and [C004] can now be written as

$$\frac{\partial \mathcal{L}_z}{\partial D_z} = PL_z - \mu_z A_z^{-\rho_z} \alpha_z \left( \frac{Q_z}{D_z} \right)^{\rho_z+1} = 0 \quad [C015]$$

$$\frac{\partial \mathcal{L}_z}{\partial IMT_z} = PMT_z - \mu_z A_z^{-\rho_z} (1 - \alpha_z) \left( \frac{Q_z}{IMT_z} \right)^{\rho_z+1} = 0 \quad [C016]$$

or

$$PL_z = \mu_z A_z^{-\rho_z} \alpha_z \left( \frac{Q_z}{D_z} \right)^{\rho_z+1} \quad [C017]$$

$$PMT_z = \mu_z A_z^{-\rho_z} (1 - \alpha_z) \left( \frac{Q_z}{IMT_z} \right)^{\rho_z+1} \quad [C018]$$

### C.3 Relative demand for domestic production and imports

Take the ratio of [C017] and [C018]

$$\frac{PL_z}{PMT_z} = \frac{\mu_z A_z^{-\rho_z} \alpha_z \left( \frac{Q_z}{D_z} \right)^{\rho_z+1}}{\mu_z A_z^{-\rho_z} (1 - \alpha_z) \left( \frac{Q_z}{IMT_z} \right)^{\rho_z+1}} \quad [C019]$$

$$\frac{PL_z}{PMT_z} = \frac{\alpha_z IMT_z^{\rho_z+1}}{(1 - \alpha_z) D_z^{\rho_z+1}} \quad [C020]$$

$$\left( \frac{IMT_z}{D_z} \right)^{\rho_z+1} = \frac{(1 - \alpha_z) PL_z}{\alpha_z PMT_z} \quad [C021]$$

$$\frac{IMT_z}{D_z} = \left( \frac{(1 - \alpha_z) PL_z}{\alpha_z PMT_z} \right)^{\frac{1}{\rho_z+1}} \quad [C022]$$

where  $\rho_z = \frac{1 - \sigma_z}{\sigma_z}$  implies  $\rho_z + 1 = \frac{1}{\sigma_z}$ , and

$$\frac{IMT_z}{D_z} = \left( \frac{(1 - \alpha_z)PL_z}{\alpha_z PMT_z} \right)^{\sigma_z} \quad [C023]$$

$$IMT_z = \left( \frac{(1 - \alpha_z)PL_z}{\alpha_z PMT_z} \right)^{\sigma_z} D_z \quad [C024]$$

#### C.4 Demand for components in terms of composite demand and prices

Substitute [C024] into [009]

$$Q_z = A_z \left\{ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) \left[ \left( \frac{(1 - \alpha_z)PL_z}{\alpha_z PMT_z} \right)^{\sigma_z} D_z \right]^{-\rho_z} \right\}^{-\frac{1}{\rho_z}} \quad [C025]$$

$$Q_z = A_z D_z \left\{ \alpha_z + (1 - \alpha_z) \left[ \left( \frac{(1 - \alpha_z)PL_z}{\alpha_z PMT_z} \right)^{\sigma_z} \right]^{-\rho_z} \right\}^{-\frac{1}{\rho_z}} \quad [C026]$$

where  $\rho_z = \frac{1 - \sigma_z}{\sigma_z}$  implies  $\sigma_z = \frac{1}{\rho_z + 1}$ , and

$$Q_z = A_z D_z \left\{ \alpha_z + (1 - \alpha_z) \left( \frac{(1 - \alpha_z)PL_z}{\alpha_z PMT_z} \right)^{-\frac{\rho_z}{\rho_z + 1}} \right\}^{-\frac{1}{\rho_z}} \quad [C027]$$

$$\frac{D_z}{Q_z} = \frac{1}{A_z} \left\{ \alpha_z + (1 - \alpha_z) \left( \frac{(1 - \alpha_z)PL_z}{\alpha_z PMT_z} \right)^{-\frac{\rho_z}{\rho_z + 1}} \right\}^{\frac{1}{\rho_z}} \quad [C028]$$

$$\frac{D_z}{Q_z} = \frac{1}{A_z} \left\{ \left( \frac{\alpha_z}{PL_z} \right)^{\frac{\rho_z}{\rho_z + 1}} \left[ \alpha_z \left( \frac{\alpha_z}{PL_z} \right)^{-\frac{\rho_z}{\rho_z + 1}} + (1 - \alpha_z) \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{-\frac{\rho_z}{\rho_z + 1}} \right] \right\}^{\frac{1}{\rho_z}} \quad [C029]$$

$$\frac{D_z}{Q_z} = \frac{1}{A_z} \left( \frac{\alpha_z}{PL_z} \right)^{\frac{1}{\rho_z+1}} \left[ \alpha_z \left( \frac{\alpha_z}{PL_z} \right)^{-\frac{\rho_z}{\rho_z+1}} + (1 - \alpha_z) \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{-\frac{\rho_z}{\rho_z+1}} \right]^{\frac{1}{\rho_z}} \quad [C030]$$

where  $\rho_z = \frac{1 - \sigma_z}{\sigma_z}$  implies  $\rho_z + 1 = \frac{1}{\sigma_z}$ , and  $\frac{\rho_z}{\rho_z + 1} = 1 - \sigma_z$ . It follows that

$$\frac{D_z}{Q_z} = \frac{1}{A_z} \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} \left[ \alpha_z \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z - 1} + (1 - \alpha_z) \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{\sigma_z - 1} \right]^{-\frac{\sigma_z}{\sigma_z - 1}} \quad [C031]$$

$$\frac{D_z}{Q_z} = \frac{1}{A_z} \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} \left[ PL_z \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} + PMT_z \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{\sigma_z} \right]^{-\frac{\sigma_z}{\sigma_z - 1}} \quad [C032]$$

where the left-hand side ratio depends only on component prices. As a matter of fact, multiplying both sides of [C032] by  $Q_z$  transforms it into a demand equation of  $D_z$ . A similar development would lead to a demand equation for  $IMT_z$ .

$$\frac{IMT_z}{Q_z} = \frac{1}{A_z} \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{\sigma_z} \left[ PL_z \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} + PMT_z \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{\sigma_z} \right]^{-\frac{\sigma_z}{\sigma_z - 1}} \quad [C033]$$

### C.5 Unit cost of composite demand

It is clear from [C023] that the relative demand for domestic production and imports is independent of the scale of demand, consistent with the first-degree homogeneity of aggregator function [009]. Consequently, the unit cost of the composite demand can be obtained from [015]

$$PC_z Q_z = PL_z D_z + PMT_z IMT_z \quad [015]$$

by substituting the optimal ratio determined in [C023], and dividing through by  $Q_z$ .

$$PC_z Q_z = \left( PL_z + PMT_z \frac{IMT_z}{D_z} \right) D_z \quad [C034]$$

$$PC_z Q_z = \left[ PL_z + PMT_z \left( \frac{(1 - \alpha_z) PL_z}{\alpha_z PMT_z} \right)^{\sigma_z} \right] D_z \quad [C035]$$

$$PC_z = \left[ PL_z + PMT_z \left( \frac{(1 - \alpha_z) PL_z}{\alpha_z PMT_z} \right)^{\sigma_z} \right] \frac{D_z}{Q_z} \quad [C036]$$

$$PC_z = \left( \frac{PL_z}{\alpha_z} \right)^{\sigma_z} \left[ PL_z \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} + PMT_z \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{\sigma_z} \right] \frac{D_z}{Q_z} \quad [C037]$$

To express  $PC_z$  in terms of the component prices only, we must substitute for  $D_z/Q_z$  in [C037] using [C032]:

$$PC_z = \left( \frac{PL_z}{\alpha_z} \right)^{\sigma_z} \left[ PL_z \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} + PMT_z \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{\sigma_z} \right] \frac{1}{A_z} \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} \left[ PL_z \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} + PMT_z \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{\sigma_z} \right]^{\frac{\sigma_z}{\sigma_z - 1}} \quad [C038]$$

$$PC_z = \frac{1}{A_z} \left[ PL_z \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} + PMT_z \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{\sigma_z} \right]^{1 + \frac{\sigma_z}{\sigma_z - 1}} \quad [C039]$$

$$PC_z = \frac{1}{A_z} \left[ PL_z \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} + PMT_z \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{\sigma_z} \right]^{-\frac{1}{\sigma_z - 1}} \quad [C040]$$

or

$$PC_z = \frac{1}{A_z} \left[ \alpha_z \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z - 1} + (1 - \alpha_z) \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{\sigma_z - 1} \right]^{-\frac{1}{\sigma_z - 1}} \quad [C041]$$

### C.6 Demand for domestic production and imports reformulated

Given [C040], we have

$$A_z PC_z = \left[ PL_z \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} + PMT_z \left( \frac{(1 - \alpha_z)}{PMT_z} \right)^{\sigma_z} \right]^{-\frac{1}{\sigma_z - 1}} \quad [C042]$$

Using [C042], we can write [C032] as

$$\frac{D_z}{Q_z} = \frac{1}{A_z} \left( \frac{\alpha_z}{PL_z} \right)^{\sigma_z} (A_z PC_z)^{\sigma_z} \quad [C043]$$

$$\frac{D_z}{Q_z} = (A_z)^{\sigma_z - 1} \left( \frac{\alpha_z PC_z}{PL_z} \right)^{\sigma_z} \quad [C044]$$

$$D_z = (A_z)^{\sigma_z - 1} \left( \frac{\alpha_z PC_z}{PL_z} \right)^{\sigma_z} Q_z \quad [C045]$$

which is an alternate form of the demand equation for domestic production. A similar development leads to the import demand equation:

$$IMT_z = (A_z)^{\sigma_z - 1} \left( \frac{(1 - \alpha_z) PC_z}{PMT_z} \right)^{\sigma_z} Q_z \quad [C046]$$

Equation [025] defines  $D_z^D$  from [C045].



## APPENDIX D: EQUATION [026]

### D.1 Sales revenue maximizing problem

The regional agent allocates production between the domestic market and exports by maximizing the value of production

$$P_z X S_z = P L_z D_z + P X T_z E X T_z \quad [013]$$

subject to

$$X S_z = B_z \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) E X T_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} \text{ where } \kappa_z = \frac{\tau_z + 1}{\tau_z}, \text{ with } 0 < \tau_z < \infty \quad [005]$$

### D.2 Lagrangian and first-order conditions

Form the Lagrangian

$$\mathcal{L}_z = P L_z D_z + P X T_z E X T_z - \lambda_z \left\{ B_z \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) E X T_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} - X S_z \right\} \quad [D001]$$

The first order conditions are:

$$\frac{\partial \mathcal{L}_z}{\partial \lambda_z} = -B_z \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) E X T_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} + X S_z = 0 \quad [D002]$$

$$\frac{\partial \mathcal{L}_z}{\partial D_z} = P L_z - \lambda_z \frac{\partial}{\partial D_z} B_z \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) E X T_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} = 0 \quad [D003]$$

$$\frac{\partial \mathcal{L}_z}{\partial E X T_z} = P X T_z - \lambda_z \frac{\partial}{\partial E X T_z} B_z \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) E X T_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} = 0 \quad [D004]$$

with

$$\frac{\partial}{\partial D_z} B_z \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) E X T_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} = \frac{B_z}{\kappa_z} \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) E X T_z^{\kappa_z} \right]^{\frac{1}{\kappa_z} - 1} \frac{\partial}{\partial D_z} \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) E X T_z^{\kappa_z} \right] \quad [D005]$$

$$\frac{\partial}{\partial D_z} B_z [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z}} =$$

$$B_z [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z} - 1} \beta_z D_z^{\kappa_z - 1}$$
[D006]

and

$$\frac{\partial}{\partial EXT_z} B_z [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z}} =$$

$$\frac{B_z}{\kappa_z} [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z} - 1} \frac{\partial}{\partial EXT_z} [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]$$
[D007]

$$\frac{\partial}{\partial EXT_z} B_z [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z}} =$$

$$B_z [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z} - 1} (1 - \beta_z) EXT_z^{\kappa_z - 1}$$
[D008]

Given [005],

$$\frac{XS_z}{B_z} = [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z}}$$
[D009]

$$\left( \frac{XS_z}{B_z} \right)^{\kappa_z} = [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]$$
[D010]

Substitute [D010] into [D006] to obtain

$$\frac{\partial}{\partial D_z} B_z [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z}} = B_z \left[ \left( \frac{XS_z}{B_z} \right)^{\kappa_z} \right]^{\frac{1}{\kappa_z} - 1} \beta_z D_z^{\kappa_z - 1}$$
[D011]

$$\frac{\partial}{\partial D_z} B_z [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z}} = B_z^{\kappa_z} XS_z^{1 - \kappa_z} \beta_z D_z^{\kappa_z - 1}$$
[D012]

$$\frac{\partial}{\partial D_z} B_z [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z}} = B_z^{\kappa_z} \beta_z \left( \frac{D_z}{XS_z} \right)^{\kappa_z - 1}$$
[D013]

and similarly into [D008], yielding

$$\frac{\partial}{\partial EXT_z} B_z [\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z}]^{\frac{1}{\kappa_z}} = B_z^{\kappa_z} (1 - \beta_z) \left( \frac{EXT_z}{XS_z} \right)^{\kappa_z - 1} \quad [D014]$$

First-order conditions [D003] and [D004] can now be written as

$$\frac{\partial \mathcal{L}_z}{\partial D_z} = PL_z - \lambda_z B_z^{\kappa_z} \beta_z \left( \frac{D_z}{XS_z} \right)^{\kappa_z - 1} = 0 \quad [D015]$$

$$\frac{\partial \mathcal{L}_z}{\partial EXT_z} = PXT_z - \lambda_z B_z^{\kappa_z} (1 - \beta_z) \left( \frac{EXT_z}{XS_z} \right)^{\kappa_z - 1} = 0 \quad [D016]$$

or

$$PL_z = \lambda_z B_z^{\kappa_z} \beta_z \left( \frac{D_z}{XS_z} \right)^{\kappa_z - 1} \quad [D017]$$

$$PXT_z = \lambda_z B_z^{\kappa_z} (1 - \beta_z) \left( \frac{EXT_z}{XS_z} \right)^{\kappa_z - 1} \quad [D018]$$

### D.3 Relative supply of components

Take the ratio of [D017] and [D018]

$$\frac{PL_z}{PXT_z} = \frac{\lambda_z B_z^{\kappa_z} \beta_z \left( \frac{D_z}{XS_z} \right)^{\kappa_z - 1}}{\lambda_z B_z^{\kappa_z} (1 - \beta_z) \left( \frac{EXT_z}{XS_z} \right)^{\kappa_z - 1}} \quad [D019]$$

$$\frac{PL_z}{PXT_z} = \frac{\beta_z (D_z)^{\kappa_z - 1}}{(1 - \beta_z) (EXT_z)^{\kappa_z - 1}} \quad [D020]$$

$$\frac{(D_z)^{\kappa_z - 1}}{(EXT_z)^{\kappa_z - 1}} = \frac{(1 - \beta_z) PL_z}{\beta_z PXT_z} \quad [D021]$$

$$\frac{D_z}{EXT_z} = \left( \frac{(1 - \beta_z) PL_z}{\beta_z PXT_z} \right)^{\frac{1}{\kappa_z - 1}} \quad [D022]$$

where  $\kappa_z = \frac{\tau_z + 1}{\tau_z}$  implies  $\kappa_z - 1 = \frac{1}{\tau_z}$  and

$$\frac{D_z}{EXT_z} = \left( \frac{(1 - \beta_z) PL_z}{\beta_z PXT_z} \right)^{\tau_z} \quad [D023]$$

$$D_z = \left( \frac{(1 - \beta_z) PL_z}{\beta_z PXT_z} \right)^{\tau_z} EXT_z \quad [D024]$$

#### D.4 Supply of components in terms of aggregate production and prices

Substitute [D024] into [005]

$$XS_z = B_z \left\{ \beta_z \left[ \left( \frac{(1 - \beta_z) PL_z}{\beta_z PXT_z} \right)^{\tau_z} EXT_z \right]^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z} \right\}^{\frac{1}{\kappa_z}} \quad [D025]$$

$$XS_z = B_z EXT_z \left\{ \beta_z \left[ \left( \frac{(1 - \beta_z) PL_z}{\beta_z PXT_z} \right)^{\tau_z} \right]^{\kappa_z} + (1 - \beta_z) \right\}^{\frac{1}{\kappa_z}} \quad [D026]$$

where  $\kappa_z = \frac{\tau_z + 1}{\tau_z}$  implies  $\tau_z = \frac{1}{\kappa_z - 1}$  and

$$XS_z = B_z EXT_z \left\{ \beta_z \left[ \left( \frac{(1 - \beta_z) PL_z}{\beta_z PXT_z} \right)^{\frac{1}{\kappa_z - 1}} \right]^{\kappa_z} + (1 - \beta_z) \right\}^{\frac{1}{\kappa_z}} \quad [D027]$$

$$XS_z = B_z EXT_z \left\{ \beta_z \left( \frac{(1 - \beta_z) PL_z}{\beta_z PXT_z} \right)^{\frac{\kappa_z}{\kappa_z - 1}} + (1 - \beta_z) \right\}^{\frac{1}{\kappa_z}} \quad [D028]$$

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z} \left\{ \beta_z \left( \frac{(1 - \beta_z) PL_z}{\beta_z PXT_z} \right)^{\frac{\kappa_z}{\kappa_z - 1}} + (1 - \beta_z) \right\}^{-\frac{1}{\kappa_z}} \quad [D029]$$

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z} \left\{ \left( \frac{(1 - \beta_z)}{PXT_z} \right)^{\frac{\kappa_z}{\kappa_z - 1}} \left[ \beta_z \left( \frac{PL_z}{\beta_z} \right)^{\frac{\kappa_z}{\kappa_z - 1}} + (1 - \beta_z) \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\frac{\kappa_z}{\kappa_z - 1}} \right] \right\}^{-\frac{1}{\kappa_z}} \quad [D030]$$

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z} \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\frac{1}{\kappa_z - 1}} \left[ \beta_z \left( \frac{PL_z}{\beta_z} \right)^{\frac{\kappa_z}{\kappa_z - 1}} + (1 - \beta_z) \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\frac{\kappa_z}{\kappa_z - 1}} \right]^{-\frac{1}{\kappa_z}} \quad [D031]$$

where  $\kappa_z = \frac{\tau_z + 1}{\tau_z}$  implies  $\kappa_z - 1 = \frac{1}{\tau_z}$  and  $\frac{\kappa_z}{\kappa_z - 1} = \tau_z + 1$ . It follows that

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z} \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \left[ \beta_z \left( \frac{PL_z}{\beta_z} \right)^{\tau_z + 1} + (1 - \beta_z) \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z + 1} \right]^{-\frac{\tau_z}{\tau_z + 1}} \quad [D032]$$

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z} \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \left[ PL_z \left( \frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \right]^{-\frac{\tau_z}{\tau_z + 1}} \quad [D033]$$

where the left-hand side ratio depends only on component prices. As a matter of fact, multiplying both sides of [D033] by  $XS_z$  transforms it into a supply equation of  $EXT_z$ . A similar development would lead to a supply equation for  $D_z$ .

$$\frac{D_z}{XS_z} = \frac{1}{B_z} \left( \frac{PL_z}{\beta_z} \right)^{\tau_z} \left[ PL_z \left( \frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \right]^{-\frac{\tau_z}{\tau_z + 1}} \quad [D034]$$

### D.5 Unit value of aggregate output

It is clear from [D023] that the relative supply on the domestic and export markets is independent of the scale of output, consistent with the first-degree homogeneity of aggregator function [005]. Consequently, the unit value of the output aggregate can be obtained from [013]

$$P_z XS_z = PL_z D_z + PXT_z EXT_z \quad [013]$$

by substituting the optimal ratio determined in [D023], and dividing through by  $XS_z$ .

$$P_z XS_z = \left( PL_z \frac{D_z}{EXT_z} + PXT_z \right) EXT_z \quad [D035]$$

$$P_z XS_z = \left[ PL_z \left( \frac{(1 - \beta_z) PL_z}{\beta_z PXT_z} \right)^{\tau_z} + PXT_z \right] EXT_z \quad [D036]$$

$$P_z = \left[ PL_z \left( \frac{(1 - \beta_z) PL_z}{\beta_z PXT_z} \right)^{\tau_z} + PXT_z \right] \frac{EXT_z}{XS_z} \quad [D037]$$

$$P_z = \left( \frac{(1 - \beta_z)}{PXT_z} \right)^{\tau_z} \left[ PL_z \left( \frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \right] \frac{EXT_z}{XS_z} \quad [D038]$$

To express  $P_z$  in terms of the component prices only, we must substitute for  $EXT_z/XS_z$  in [D038] using [D033]:

$$P_z = \left( \frac{(1 - \beta_z)}{PXT_z} \right)^{\tau_z} \left[ PL_z \left( \frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \right] \frac{1}{B_z} \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \left[ PL_z \left( \frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \right]^{-\frac{\tau_z}{\tau_z + 1}} \quad [D039]$$

$$P_z = \frac{1}{B_z} \left[ PL_z \left( \frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \right] \left[ PL_z \left( \frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \right]^{-\frac{\tau_z}{\tau_z + 1}} \quad [D040]$$

$$P_z = \frac{1}{B_z} \left[ PL_z \left( \frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \right]^{\frac{1}{\tau_z + 1}} \quad [D041]$$

or

$$P_z = \frac{1}{B_z} \left[ \beta_z \left( \frac{PL_z}{\beta_z} \right)^{\tau_z + 1} + (1 - \beta_z) \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z + 1} \right]^{\frac{1}{\tau_z + 1}} \quad [D042]$$

### D.6 Supply of components reformulated

Given [D041], we have

$$B_z P_z = \left[ PL_z \left( \frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} \right]^{\frac{1}{\tau_z + 1}} \quad [D043]$$

Using [D043], we can write [D033] as

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z} \left( \frac{PXT_z}{(1 - \beta_z)} \right)^{\tau_z} (B_z P_z)^{-\tau_z} \quad [D044]$$

$$\frac{EXT_z}{XS_z} = \left( \frac{1}{B_z} \right)^{\tau_z + 1} \left( \frac{PXT_z}{(1 - \beta_z) P_z} \right)^{\tau_z} \quad [D045]$$

$$EXT_z = \left( \frac{1}{B_z} \right)^{\tau_z + 1} \left( \frac{PXT_z}{(1 - \beta_z) P_z} \right)^{\tau_z} XS_z \quad [D046]$$

which is an alternate form of the export supply equation. A similar development leads to the domestic supply equation:

$$D_z = \left( \frac{1}{B_z} \right)^{\tau_z + 1} \left( \frac{PL_z}{\beta_z P_z} \right)^{\tau_z} XS_z \quad [D047]$$

Equation [026] defines  $D_z^O$  from [D047].



## APPENDIX E: EQUATION [027]

### E.1 Cost minimizing problem

The regional agent allocates imports between origins by minimizing the aggregate cost of imports

$$PMT_z IMT_z = e_z \sum_{zj} PW_{zj,z} IM_{zj,z} \quad [016]$$

subject to

$$IMT_z = A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} \quad \text{where } \rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M}, \text{ with } 0 < \sigma_z^M < \infty \quad [011]$$

$$IMT_z = IMT_z^* \quad [E001]$$

where  $IMT_z^*$  is the solution to the demand allocation problem as given by [C046].

### E.2 Lagrangian and first-order conditions

Write the Lagrangian

$$\mathcal{L}_z = \sum_{zj} PW_{zj,z} IM_{zj,z} - \mu_z^M \left\{ A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} - IMT_z^* \right\} \quad [E002]$$

The first order conditions are:

$$\frac{\partial \mathcal{L}_z}{\partial \mu_z^M} = -A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} + IMT_z^* = 0 \quad [E003]$$

$$\frac{\partial \mathcal{L}_z}{\partial IM_{zj,z}} = PW_{zj,z} - \mu_z^M \frac{\partial}{\partial IM_{zj,z}} A_z^M \left[ \sum_{zjj} \alpha_{zjj,z}^M IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} = 0 \quad [E004]$$

where

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[ \sum_{zjj} \alpha_{zjj,z}^M IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} =$$

$$- A_z^M \frac{1}{\rho_z^M} \left[ \sum_{zjj} \alpha_{zjj,z}^M IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M} - 1} \frac{\partial}{\partial IM_{zj,z}} \left[ \sum_{zjj} \alpha_{zjj,z}^M IM_{zjj,z}^{-\rho_z^M} \right]$$
[E005]

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[ \sum_{zjj} \alpha_{zjj,z}^M IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} =$$

$$- A_z^M \frac{1}{\rho_z^M} \left[ \sum_{zjj} \alpha_{zjj,z}^M IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M} - 1} \alpha_{zj,z}^M (-\rho_z^M) IM_{zj,z}^{-\rho_z^M - 1}$$
[E006]

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[ \sum_{zjj} \alpha_{zjj,z}^M IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} = A_z^M \left[ \sum_{zjj} \alpha_{zjj,z}^M IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M} - 1} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M - 1}$$
[E007]

Given [011]

$$IMT_z = A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}}$$
[011]

which is equivalent to

$$\left( \frac{IMT_z}{A_z^M} \right)^{-\rho_z^M} = \left[ \sum_{zj} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M} \right]$$
[E008]

we can write

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[ \sum_{zjj} \alpha_{zjj,z}^M IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} = A_z^M \left[ \left( \frac{IMT_z^*}{A_z^M} \right)^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M} - 1} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M - 1}$$
[E009]

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[ \sum_{zjj} \alpha_{zjj,z}^M IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} = A_z^M \left( \frac{IMT_z^*}{A_z^M} \right)^{\rho_z^M + 1} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M - 1}$$
[E010]

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[ \sum_{zjj} \alpha_{zjj,z}^M IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} = (A_z^M)^{-\rho_z^M} \alpha_{zj,z}^M \left( \frac{IMT_z^*}{IM_{zj,z}} \right)^{\rho_z^M + 1} \quad [E011]$$

And the first-order condition becomes

$$\frac{\partial \mathcal{L}_z}{\partial IM_{zj,z}} = PW_{zj,z} - \mu_z^M (A_z^M)^{-\rho_z^M} \alpha_{zj,z}^M \left( \frac{IMT_z^*}{IM_{zj,z}} \right)^{\rho_z^M + 1} = 0 \quad [E012]$$

or

$$PW_{zj,z} = \mu_z^M (A_z^M)^{-\rho_z^M} \alpha_{zj,z}^M \left( \frac{IMT_z^*}{IM_{zj,z}} \right)^{\rho_z^M + 1} \quad [E013]$$

### E.3 Relative demand for imports of different origins

Take the ratio of [aa158] for two different origins,  $zj$  and  $zjj$ :

$$\frac{PW_{zjj,z}}{PW_{zj,z}} = \frac{\mu_z^M (A_z^M)^{-\rho_z^M} \alpha_{zjj,z}^M \left( \frac{IMT_z^*}{IM_{zjj,z}} \right)^{\rho_z^M + 1}}{\mu_z^M (A_z^M)^{-\rho_z^M} \alpha_{zj,z}^M \left( \frac{IMT_z^*}{IM_{zj,z}} \right)^{\rho_z^M + 1}} \quad [E014]$$

$$\frac{PW_{zjj,z}}{PW_{zj,z}} = \frac{\alpha_{zjj,z}^M \left( \frac{IM_{zj,z}}{IM_{zjj,z}} \right)^{\rho_z^M + 1}}{\alpha_{zj,z}^M} \quad [E015]$$

$$\frac{IM_{zj,z}}{IM_{zjj,z}} = \left( \frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{\frac{1}{\rho_z^M + 1}} \quad [E016]$$

or, given  $\rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M}$  and  $\rho_z^M + 1 = \frac{1}{\sigma_z^M}$ ,

$$\frac{IM_{zj,z}}{IM_{zjj,z}} = \left( \frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{\sigma_z^M} \quad [E017]$$

$$IM_{zj,z} = \left( \frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{\sigma_z^M} IM_{zjj,z} \quad [E018]$$

#### E.4 Demand for components in terms of aggregate imports and prices

Substitute [E018] into [011]

$$IMT_z^* = A_z^M \left[ \sum_{zj} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} \quad [E011]$$

$$IMT_z^* = A_z^M \left\{ \sum_{zj} \alpha_{zj,z}^M \left[ \left( \frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{\sigma_z^M} IM_{zjj,z} \right]^{-\rho_z^M} \right\}^{\frac{-1}{\rho_z^M}} \quad [E019]$$

where  $\rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M}$  implies  $\sigma_z^M = \frac{1}{\rho_z^M + 1}$ , and

$$IMT_z^* = A_z^M \left\{ \sum_{zj} \alpha_{zj,z}^M \left[ \left( \frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{\frac{1}{\rho_z^M + 1}} IM_{zjj,z} \right]^{-\rho_z^M} \right\}^{\frac{-1}{\rho_z^M}} \quad [E020]$$

$$IMT_z^* = A_z^M IM_{zjj,z} \left\{ \sum_{zj} \alpha_{zj,z}^M \left[ \left( \frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{\frac{1}{\rho_z^M + 1}} \right]^{-\rho_z^M} \right\}^{\frac{-1}{\rho_z^M}} \quad [E021]$$

$$IMT_z^* = A_z^M IM_{zjj,z} \left[ \sum_{zj} \alpha_{zj,z}^M \left( \frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{-\frac{\rho_z^M}{\rho_z^M + 1}} \right]^{\frac{-1}{\rho_z^M}} \quad [E022]$$

$$IMT_z^* = A_z^M IM_{zjj,z} \left( \frac{PW_{zjj,z}}{\alpha_{zjj,z}^M} \right)^{\frac{1}{\rho_z^M + 1}} \left[ \sum_{zj} \alpha_{zj,z}^M \left( \frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{-\frac{\rho_z^M}{\rho_z^M + 1}} \right]^{\frac{-1}{\rho_z^M}} \quad [E023]$$

$$\frac{IM_{zjj,z}}{IMT_z^*} = \frac{1}{A_z^M} \left( \frac{PW_{zjj,z}}{\alpha_{zjj,z}^M} \right)^{-\frac{1}{\rho_z^M + 1}} \left[ \sum_{zj} \alpha_{zj,z}^M \left( \frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{-\frac{\rho_z^M}{\rho_z^M + 1}} \right]^{\frac{1}{\rho_z^M}} \quad [E024]$$

where  $\rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M}$  implies  $\rho_z^M + 1 = \frac{1}{\sigma_z^M}$ , and  $\frac{\rho_z^M}{\rho_z^M + 1} = 1 - \sigma_z^M$ . Therefore

$$\frac{IM_{zjj,z}}{IMT_z^*} = \frac{1}{A_z^M} \left( \frac{PW_{zjj,z}}{\alpha_{zjj,z}^M} \right)^{-\sigma_z^M} \left[ \sum_{zj} \alpha_{zj,z}^M \left( \frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{\sigma_z^M - 1} \right]^{\frac{-\sigma_z^M}{\sigma_z^M - 1}} \quad [E025]$$

$$\frac{IM_{zjj,z}}{IMT_z^*} = \frac{1}{A_z^M} \left( \frac{PW_{zjj,z}}{\alpha_{zjj,z}^M} \right)^{-\sigma_z^M} \left[ \sum_{zj} PW_{zj,z} \left( \frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{\sigma_z^M} \right]^{\frac{-\sigma_z^M}{\sigma_z^M - 1}} \quad [E026]$$

where the left-hand side ratio depends only on component prices. As a matter of fact, multiplying both sides of [E026] by  $IMT_z^*$  transforms it into a supply equation of  $IM_{zj,z}$ .

### E.5 Unit value of aggregate imports

It is clear from [E017] that the relative demand for imports of different origins is independent of the scale of imports, consistent with the first-degree homogeneity of aggregator function [007]. Consequently, the unit value of the import aggregate can be obtained from [016]

$$PMT_z IMT_z = e_z \sum_{zj} PW_{zj,z} IM_{zj,z} \quad [016]$$

by substituting the optimal ratio determined in [E017], and dividing through by  $EXT_z^*$ .

$$PMT_z \text{ } IMT_z^* = e_z IM_{zjj,z} \sum_{zj} PW_{zj,z} \left( \frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{\sigma_z^M} \quad [\text{E027}]$$

$$PMT_z = e_z \frac{IM_{zjj,z}}{IMT_z^*} \sum_{zj} PW_{zj,z} \left( \frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{\sigma_z^M} \quad [\text{E028}]$$

$$PMT_z = e_z \frac{IM_{zjj,z}}{IMT_z^*} \left( \frac{PW_{zjj,z}}{\alpha_{zjj,z}^M} \right)^{\sigma_z^M} \sum_{zj} PW_{zj,z} \left( \frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{\sigma_z^M} \quad [\text{E029}]$$

To express  $PMT_z$  in terms of the component prices only, we must substitute for  $IM_{zjj,z}/IMT_z^*$  in [E029] using [E026]:

$$PMT_z = e_z \frac{1}{A_z^M} \left( \frac{PW_{zjj,z}}{\alpha_{zjj,z}^M} \right)^{-\sigma_z^M} \left[ \sum_{zj} PW_{zj,z} \left( \frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{\sigma_z^M} \right]^{-\frac{\sigma_z^M}{\sigma_z^M - 1}} \quad [\text{E030}]$$

$$PMT_z = e_z \frac{1}{A_z^M} \left( \frac{PW_{zjj,z}}{\alpha_{zjj,z}^M} \right)^{\sigma_z^M} \sum_{zj} PW_{zj,z} \left( \frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{\sigma_z^M} \left[ \sum_{zj} PW_{zj,z} \left( \frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{\sigma_z^M} \right]^{1 - \frac{\sigma_z^M}{\sigma_z^M - 1}} \quad [\text{E031}]$$

$$PMT_z = e_z \frac{1}{A_z^M} \left[ \sum_{zj} PW_{zj,z} \left( \frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{\sigma_z^M} \right]^{-\frac{1}{\sigma_z^M - 1}} \quad [\text{E032}]$$

### E.6 Supply of components reformulated

Given [E031], we have

$$A_z^M PMT_z = e_z \left[ \sum_{zj} PW_{zj,z} \left( \frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{\sigma_z^M} \right]^{-\frac{1}{\sigma_z^M - 1}} \quad [\text{E033}]$$

Using [E033], we can rewrite [E026] as

$$\frac{IM_{zjj,z}}{IMT_z^*} = \frac{1}{A_z^M} \left( \frac{PW_{zjj,z}}{\alpha_{zjj,z}^M} \right)^{-\sigma_z^M} (A_z^M PMT_z)^{\sigma_z^M} \quad [E034]$$

$$\frac{IM_{zjj,z}}{IMT_z^*} = \left( \frac{1}{A_z^M} \right)^{1-\sigma_z^M} \left( \frac{PW_{zjj,z}}{\alpha_{zjj,z}^M PMT_z} \right)^{-\sigma_z^M} \quad [E035]$$

$$IM_{zjj,z} = \left( \frac{1}{A_z^M} \right)^{1-\sigma_z^M} \left( \frac{PW_{zjj,z}}{\alpha_{zjj,z}^M PMT_z} \right)^{-\sigma_z^M} IMT_z^* \quad [E036]$$

which is an alternate form of the demand equation for imports from a particular region.

To obtain equation [027], substitute

$$IMT_z = (A_z)^{\sigma_z - 1} \left( \frac{(1 - \alpha_z) PC_z}{PMT_z} \right)^{\sigma_z} Q_z \quad [C046]$$

into [E036].



## APPENDIX F: EQUATION [028]

### F.1 Revenue maximizing problem

The regional agent allocates exports between export destinations by maximizing the value of exports

$$PXT_z \text{ EXT}_z = e_z \sum_{zj} PW_{z,zj} EX_{z,zj} \quad [014]$$

subject to

$$\text{EXT}_z = B_z^X \left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} \quad \text{where } \kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}, \text{ with } 0 < \tau_z^X < \infty \quad [007]$$

$$\text{EXT}_z = \text{EXT}_z^*$$

where  $\text{EXT}_z^*$  is the solution to the output allocation problem as given by equation [D046].

$$\text{EXT}_z = \left( \frac{1}{B_z} \right)^{\tau_z + 1} \left( \frac{PXT_z}{(1 - \beta_z)P_z} \right)^{\tau_z} XS_z \quad [D046]$$

### F.2 Lagrangian and first-order conditions

Write the Lagrangian

$$\mathcal{L}_z = \sum_{zj} PW_{z,zj} EX_{z,zj} - \lambda_z^X \left\{ B_z^X \left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} - \text{EXT}_z^* \right\} \quad [F001]$$

The first order conditions are:

$$\frac{\partial \mathcal{L}_z}{\partial \lambda_z^X} = -B_z^X \left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} + \text{EXT}_z^* = 0 \quad [F002]$$

$$\frac{\partial \mathcal{L}_z}{\partial EX_{z,zj}} = PW_{z,zj} - \lambda_z^X \frac{\partial}{\partial EX_{z,zj}} B_z^X \left[ \sum_{zjj} \beta_{z,zjj}^X (EX_{z,zjj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} = 0 \quad [F003]$$

where

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[ \sum_{zjj} \beta_{z,zjj}^X (EX_{z,zjj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} =$$

$$B_z^X \frac{1}{\kappa_z^X} \left[ \sum_{zjj} \beta_{z,zjj}^X (EX_{z,zjj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}-1} \frac{\partial}{\partial EX_{z,zj}} \left[ \sum_{zjj} \beta_{z,zjj}^X (EX_{z,zjj})^{\kappa_z^X} \right]$$
[F004]

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[ \sum_{zjj} \beta_{z,zjj}^X (EX_{z,zjj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} =$$

$$B_z^X \frac{1}{\kappa_z^X} \left[ \sum_{zjj} \beta_{z,zjj}^X (EX_{z,zjj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}-1} \beta_{z,zj}^X \kappa_z^X (EX_{z,zj})^{\kappa_z^X-1}$$
[F005]

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[ \sum_{zjj} \beta_{z,zjj}^X (EX_{z,zjj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} =$$

$$B_z^X \left[ \sum_{zjj} \beta_{z,zjj}^X (EX_{z,zjj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}-1} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X-1}$$
[F006]

Given [007]

$$EXT_z = B_z^X \left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}}$$
[007]

which is equivalent to

$$\left( \frac{EXT_z}{B_z^X} \right)^{\kappa_z} = \left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X} \right]$$
[F007]

we can write

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[ \sum_{zjj} \beta_{z,zjj}^X (EX_{z,zjj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} = B_z^X \left[ \left( \frac{EXT_z^*}{B_z^X} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}-1} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X-1}$$
[F008]

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[ \sum_{zjj} \beta_{z,zjj}^X (EX_{z,zjj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} = B_z^X \left( \frac{EXT_z^*}{B_z^X} \right)^{1-\kappa_z^X} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X-1} \quad [\text{FO09}]$$

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[ \sum_{zjj} \beta_{z,zjj}^X (EX_{z,zjj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} = (B_z^X)^{\kappa_z^X} \beta_{z,zj}^X \left( \frac{EX_{z,zj}}{EXT_z^*} \right)^{\kappa_z^X-1} \quad [\text{FO10}]$$

And the first-order condition becomes

$$\frac{\partial \mathcal{L}_z}{\partial EX_{z,zj}} = PW_{z,zj} - \lambda_z^X (B_z^X)^{\kappa_z^X} \beta_{z,zj}^X \left( \frac{EX_{z,zj}}{EXT_z^*} \right)^{\kappa_z^X-1} = 0 \quad [\text{FO11}]$$

or

$$PW_{z,zj} = \lambda_z^X (B_z^X)^{\kappa_z^X} \beta_{z,zj}^X \left( \frac{EX_{z,zj}}{EXT_z^*} \right)^{\kappa_z^X-1} \quad [\text{FO12}]$$

### F.3 Relative supply of components

Take the ratio of [FO12] for two different trading partners:

$$\frac{PW_{z,zj}}{PW_{z,zjj}} = \frac{\lambda_z^X (B_z^X)^{\kappa_z^X} \beta_{z,zj}^X \left( \frac{EX_{z,zj}}{EXT_z^*} \right)^{\kappa_z^X-1}}{\lambda_z^X (B_z^X)^{\kappa_z^X} \beta_{z,zjj}^X \left( \frac{EX_{z,zjj}}{EXT_z^*} \right)^{\kappa_z^X-1}} \quad [\text{FO13}]$$

$$\frac{PW_{z,zj}}{PW_{z,zjj}} = \frac{\beta_{z,zj}^X \left( \frac{EX_{z,zj}}{EXT_z^*} \right)^{\kappa_z^X-1}}{\beta_{z,zjj}^X \left( \frac{EX_{z,zjj}}{EXT_z^*} \right)^{\kappa_z^X-1}} \quad [\text{FO14}]$$

$$\left( \frac{EX_{z,zj}}{EX_{z,zjj}} \right)^{\kappa_z^X-1} = \frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}} \quad [\text{FO15}]$$

$$\frac{EX_{z,zj}}{EX_{z,zjj}} = \left( \frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}} \right)^{\frac{1}{\kappa_z^X-1}} \quad [\text{FO16}]$$

Or, given  $\kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}$  and  $\kappa_z^X - 1 = \frac{1}{\tau_z^X}$ ,

$$\frac{EX_{z,zj}}{EX_{z,zjj}} = \left( \frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}} \right)^{\tau_z^X} \quad [\text{F017}]$$

$$EX_{z,zj} = \left( \frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}} \right)^{\tau_z^X} EX_{z,zjj} \quad [\text{F018}]$$

#### F.4 Supply of components in terms of aggregate exports and prices

Substitute [F018] into [007]

$$EXT_z^* = B_z^X \left[ \sum_{zj} \beta_{z,zj}^X (EX_{z,zj})^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} \quad \text{where } \kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}, \text{ with } 0 < \tau_z^X < \infty \quad [\text{007}]$$

$$EXT_z^* = B_z^X \left\{ \sum_{zj} \beta_{z,zj}^X \left[ \left( \frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}} \right)^{\tau_z^X} EX_{z,zjj} \right]^{\kappa_z^X} \right\}^{\frac{1}{\kappa_z^X}} \quad [\text{F019}]$$

where  $\kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}$  implies  $\tau_z^X = \frac{1}{\kappa_z^X - 1}$ , and

$$EXT_z^* = B_z^X \left\{ \sum_{zj} \beta_{z,zj}^X \left[ \left( \frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}} \right)^{\frac{1}{\kappa_z^X - 1}} EX_{z,zjj} \right]^{\kappa_z^X} \right\}^{\frac{1}{\kappa_z^X}} \quad [\text{F020}]$$

$$EXT_z^* = B_z^X EX_{z,zjj} \left\{ \sum_{zj} \beta_{z,zj}^X \left( \frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}} \right)^{\frac{\kappa_z^X}{\kappa_z^X - 1}} \right\}^{\frac{1}{\kappa_z^X}} \quad [\text{F021}]$$

$$EXT_z^* = B_z^X EX_{z,zjj} \left( \frac{\beta_{z,zjj}^X}{PW_{z,zjj}} \right)^{\frac{1}{\kappa_z^X - 1}} \left\{ \sum_{zj} \beta_{z,zj}^X \left( \frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\frac{\kappa_z^X}{\kappa_z^X - 1}} \right\}^{\frac{1}{\kappa_z^X}} \quad [F022]$$

$$\frac{EX_{z,zjj}}{EXT_z^*} = \frac{1}{B_z^X} \left( \frac{PW_{z,zjj}}{\beta_{z,zjj}^X} \right)^{\frac{1}{\kappa_z^X - 1}} \left\{ \sum_{zj} \beta_{z,zj}^X \left( \frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\frac{\kappa_z^X}{\kappa_z^X - 1}} \right\}^{-\frac{1}{\kappa_z^X}} \quad [F023]$$

where  $\kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}$  implies  $\kappa_z^X - 1 = \frac{1}{\tau_z^X}$  and  $\frac{\kappa_z^X}{\kappa_z^X - 1} = \tau_z^X + 1$ . It follows that

$$\frac{EX_{z,zjj}}{EXT_z^*} = \frac{1}{B_z^X} \left( \frac{PW_{z,zjj}}{\beta_{z,zjj}^X} \right)^{\tau_z^X} \left\{ \sum_{zj} \beta_{z,zj}^X \left( \frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X + 1} \right\}^{-\frac{\tau_z^X}{\tau_z^X + 1}} \quad [F024]$$

$$\frac{EX_{z,zjj}}{EXT_z^*} = \frac{1}{B_z^X} \left( \frac{PW_{z,zjj}}{\beta_{z,zjj}^X} \right)^{\tau_z^X} \left\{ \sum_{zj} PW_{z,zj} \left( \frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X} \right\}^{-\frac{\tau_z^X}{\tau_z^X + 1}} \quad [F025]$$

where the left-hand side ratio depends only on component prices. As a matter of fact, multiplying both sides of [F025] by  $EXT_z^*$  transforms it into a supply equation of  $EX_{z,zj}$ .

### F.5 Unit value of aggregate exports

It is clear from [F017] that the relative supply on different export markets is independent of the scale of exports, consistent with the first-degree homogeneity of aggregator function [007]. Consequently, the unit value of the export aggregate can be obtained from [014]

$$PXT_z EXT_z^* = e_z \sum_{zj} PW_{z,zj} EX_{z,zj} \quad [014]$$

by substituting the optimal ratio determined in [F017], and dividing through by  $EXT_z^*$ .

$$PXT_z EXT_z^* = e_z EX_{z,zjj} \sum_{zj} PW_{z,zj} \left( \frac{EX_{z,zj}}{EX_{z,zjj}} \right) \quad [F026]$$

$$PXT_z = e_z \frac{EX_{z,zjj}}{EXT_z^*} \sum_{zj} PW_{z,zj} \left( \frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}} \right)^{\tau_z^X} \quad [F027]$$

$$PXT_z = e_z \frac{EX_{z,zjj}}{EXT_z^*} \left( \frac{\beta_{z,zjj}^X}{PW_{z,zjj}} \right)^{\tau_z^X} \sum_{zj} PW_{z,zj} \left( \frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X} \quad [F027]$$

To express  $PXT_z$  in terms of the component prices only, we must substitute for  $EX_{z,zjj}/EXT_z^*$  in [F027] using [F025]:

$$PXT_z = e_z \frac{1}{B_z^X} \left( \frac{PW_{z,zjj}}{\beta_{z,zjj}^X} \right)^{\tau_z^X} \left\{ \sum_{zj} PW_{z,zj} \left( \frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X} \right\}^{-\frac{\tau_z^X}{\tau_z^X+1}} \quad [F028]$$

$$PXT_z = e_z \frac{1}{B_z^X} \left( \frac{\beta_{z,zjj}^X}{PW_{z,zjj}} \right)^{\tau_z^X} \sum_{zj} PW_{z,zj} \left( \frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X} \left\{ \sum_{zj} PW_{z,zj} \left( \frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X} \right\}^{1-\frac{\tau_z^X}{\tau_z^X+1}} \quad [F029]$$

$$PXT_z = e_z \frac{1}{B_z^X} \left\{ \sum_{zj} PW_{z,zj} \left( \frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X} \right\}^{\frac{1}{\tau_z^X+1}} \quad [F030]$$

or

$$PXT_z = e_z \frac{1}{B_z^X} \left\{ \sum_{zj} \beta_{z,zj}^X \left( \frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X+1} \right\}^{\frac{1}{\tau_z^X+1}} \quad [F031]$$

### F.6 Supply of components reformulated

Given [F030], we have

$$B_z^X PXT_z = e_z \left\{ \sum_{zj} PW_{z,zj} \left( \frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X} \right\}^{\frac{1}{\tau_z^X + 1}} \quad [\text{F032}]$$

Using [F032], we can rewrite [F025] as

$$\frac{EX_{z,zjj}}{EXT_z^*} = \frac{1}{B_z^X} \left( \frac{PW_{z,zjj}}{\beta_{z,zjj}^X} \right)^{\tau_z^X} (B_z^X PXT_z)^{-\tau_z^X} \quad [\text{F033}]$$

$$\frac{EX_{z,zjj}}{EXT_z^*} = \left( \frac{1}{B_z^X} \right)^{\tau_z^X + 1} \left( \frac{PW_{z,zjj}}{\beta_{z,zjj}^X PXT_z} \right)^{\tau_z^X} \quad [\text{F034}]$$

$$EX_{z,zjj} = \left( \frac{1}{B_z^X} \right)^{\tau_z^X + 1} \left( \frac{PW_{z,zjj}}{\beta_{z,zjj}^X PXT_z} \right)^{\tau_z^X} EXT_z^* \quad [\text{F035}]$$

which is an alternate form of the supply equation of exports to a particular region.

To obtain equation [027], substitute

$$EXT_z = \left( \frac{1}{B_z} \right)^{\tau_z + 1} \left( \frac{PXT_z}{(1 - \beta_z)P_z} \right)^{\tau_z} XS_z \quad [\text{D046}]$$

into [F035]



### APPENDIX G: TWO-REGION VERSION OF MODEL 3

In a 2-region model, each region has only one trading partner, so that

$$EXT_z = EX_{z,zj} \quad [G001]$$

$$IMT_z = IM_{zj,z} \quad [G002]$$

$$PW_{zj,z} = PMT_z^* \quad [G003]$$

$$PW_{z,zj} = PXT_z^* \quad [G004]$$

replacing [007], [011], [008] and [012] respectively.

Equations [G001] and [G002], together with [017], imply

$$EXT_{zj} = IMT_z \quad [067]$$

And equations [G003] and [G004] imply

$$PXT_{zj}^* = PMT_z^* \quad [069]$$

So we can insert equations [067] and [069] into the model, and do away with equations [017], [G001], [G002], [G003] and [G004], and with variables  $EX_{z,zj}$ ,  $IM_{z,zj}$  and  $PW_{z,zj}$ . The  $PWINDEX$  variable is consequently re-defined, and equation [059] is replaced by

$$PWINDEX = \sqrt{\frac{\sum_z PMT_z IMT_z^O}{\sum_z PMT_z^O IMT_z^O} \frac{\sum_z PMT_z IMT_z}{\sum_z PMT_z^O IMT_z}} \quad [066]$$

The two-region Model 3 variables are listed in Table G1, and the equations in Table G2. This model has  $11N + 1 = 23$  variables and  $9N + (N - 1) + 1 = 20$  equations. There are  $N + 1 = 3$  degrees of freedom.

**Table G1 – Two-region Model 3 variables**

<b>Volumes</b>	
$Q_z$	Domestic demand for the composite good in region z
$D_z$	Domestic demand for the locally produced good in region z
$IMT_z$	Total imports of region z
$EXT_z$	Total exports of region z
$CABX_z$	Real current account balance (pseudo-volume variable)
<b>Prices</b>	
$P_z^*$	Producer price
$PL_z^*$	Market price of local product
$PC_z^*$	Price of the composite good
$PMT_z^*$	Price of composite imports to region z
$PXT_z^*$	Price of composite exports from region z
$PW_{z,zj}$	World price of exports from region z to region zj
<b>Nominal value variables</b>	
$CAB_z^*$	Current account balance of region z

**Table G2 – Two-region Model 3 equations**

$$CAB_z^* = P_z^* XS_z - PC_z^* Q_z, z \neq z_{\text{leon}} \quad [042]$$

$$CABX_z = \frac{CAB_z^*}{PWINDEX} \quad [057]$$

$$XS_z = B_z \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} \text{ where } \kappa_z = \frac{\tau_z + 1}{\tau_z}, \text{ with } 0 < \tau_z < \infty \quad [005]$$

$$\frac{EXT_z}{D_z} = \left( \frac{\beta_z}{1 - \beta_z} \frac{PXT_z^*}{PL_z^*} \right)^{\tau_z} \quad [006]$$

$$Q_z = A_z \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{\frac{1}{\rho_z}} \text{ where } \rho_z = \frac{1 - \sigma_z}{\sigma_z}, \text{ with } 0 < \sigma_z < \infty \quad [009]$$

$$\frac{IMT_z}{D_z} = \frac{\left( \frac{1 - \alpha_z}{PMT_z^*} \right)^{\sigma_z}}{\left( \frac{\alpha_z}{PL_z^*} \right)^{\sigma_z}} = \left( \frac{1 - \alpha_z}{\alpha_z} \frac{PL_z^*}{PMT_z^*} \right)^{\sigma_z} \quad [010]$$

$$P_z^* XS_z = PL_z^* D_z + PXT_z^* EXT_z \quad [013]$$

$$PC_z^* Q_z = PL_z^* D_z + PMT_z^* IMT_z \quad [015]$$

$$PWINDEX = \sqrt{\frac{\sum_z PMT_z IMT_z^O}{\sum_z PMT_z^O IMT_z^O} \frac{\sum_z PMT_z IMT_z}{\sum_z PMT_z^O IMT_z}} \quad [H005]$$

$$EXT_{zj} = IMT_z \quad [066]$$

$$PXT_{zj}^* = PMT_z^* \quad [068]$$



## APPENDIX H: EQUATION [068]

The 2-region Q-Model consists of

$$XS_z = B_z \left[ \beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} \text{ where } \kappa_z = \frac{\tau_z + 1}{\tau_z}, \text{ with } 0 < \tau_z < \infty \quad [005]$$

$$Q_z = A_z \left[ \alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{\frac{-1}{\rho_z}} \text{ where } \rho_z = \frac{1 - \sigma_z}{\sigma_z}, \text{ with } 0 < \sigma_z < \infty \quad [009]$$

$$EXT_{zj} = IMT_z \quad [067]$$

Develop [005] when  $XS_z = \overline{XS_z}$

$$\beta_z D_z^{\kappa_z} + (1 - \beta_z) IMT_{zj}^{\kappa_z} = \left( \frac{\overline{XS_z}}{B_z} \right)^{\kappa_z} \quad [H001]$$

$$D_z^{\kappa_z} = \frac{1}{\beta_z} \left( \frac{\overline{XS_z}}{B_z} \right)^{\kappa_z} - \frac{(1 - \beta_z)}{\beta_z} IMT_{zj}^{\kappa_z} \quad [H002]$$

$$D_z = \left[ \frac{1}{\beta_z} \left( \frac{\overline{XS_z}}{B_z} \right)^{\kappa_z} - \frac{(1 - \beta_z)}{\beta_z} IMT_{zj}^{\kappa_z} \right]^{\frac{1}{\kappa_z}} \quad [H003]$$

Next, let

$$Q_z = \overline{Q_z} \quad [H004]$$

Substitute into [009] and develop.

$$\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} = \left( \frac{\overline{Q_z}}{A_z} \right)^{-\rho_z} \quad [H005]$$

$$D_z^{-\rho_z} = \frac{1}{\alpha_z} \left( \frac{\overline{Q_z}}{A_z} \right)^{-\rho_z} - \frac{(1 - \alpha_z)}{\alpha_z} IMT_z^{-\rho_z} \quad [H006]$$

$$D_z = \left[ \frac{1}{\alpha_z} \left( \frac{\overline{Q_z}}{A_z} \right)^{-\rho_z} - \frac{(1 - \alpha_z)}{\alpha_z} IMT_z^{-\rho_z} \right]^{\frac{-1}{\rho_z}} \quad [H007]$$

Combine [H003] and [H007]

$$\left[ \frac{1}{\alpha_z} \left( \frac{\overline{Q}_z}{A_z} \right)^{-\rho_z} - \frac{(1 - \alpha_z)}{\alpha_z} IMT_z^{-\rho_z} \right]^{\frac{-1}{\rho_z}} = \left[ \frac{1}{\beta_z} \left( \frac{\overline{XS}_z}{B_z} \right)^{\kappa_z} - \frac{(1 - \beta_z)}{\beta_z} IMT_{zj}^{\kappa_z} \right]^{\frac{1}{\kappa_z}} \quad [\text{H008}]$$

$$\frac{1}{\alpha_z} \left( \frac{\overline{Q}_z}{A_z} \right)^{-\rho_z} - \frac{(1 - \alpha_z)}{\alpha_z} IMT_z^{-\rho_z} = \left[ \frac{1}{\beta_z} \left( \frac{\overline{XS}_z}{B_z} \right)^{\kappa_z} - \frac{(1 - \beta_z)}{\beta_z} IMT_{zj}^{\kappa_z} \right]^{\frac{-\rho_z}{\kappa_z}} \quad [\text{H009}]$$

$$\frac{(1 - \alpha_z)}{\alpha_z} IMT_z^{-\rho_z} = \frac{1}{\alpha_z} \left( \frac{\overline{Q}_z}{A_z} \right)^{-\rho_z} - \left[ \frac{1}{\beta_z} \left( \frac{\overline{XS}_z}{B_z} \right)^{\kappa_z} - \frac{(1 - \beta_z)}{\beta_z} IMT_{zj}^{\kappa_z} \right]^{\frac{-\rho_z}{\kappa_z}} \quad [\text{H010}]$$

$$IMT_z^{-\rho_z} = \frac{1}{(1 - \alpha_z)} \left( \frac{\overline{Q}_z}{A_z} \right)^{-\rho_z} - \frac{\alpha_z}{(1 - \alpha_z)} \left[ \frac{1}{\beta_z} \left( \frac{\overline{XS}_z}{B_z} \right)^{\kappa_z} - \frac{(1 - \beta_z)}{\beta_z} IMT_{zj}^{\kappa_z} \right]^{\frac{-\rho_z}{\kappa_z}} \quad [\text{H011}]$$

Equation [068] follows directly.

$$IMT_z =$$

$$\left\{ \frac{1}{(1 - \alpha_z)} \left( \frac{\overline{Q}_z}{A_z} \right)^{-\rho_z} - \frac{\alpha_z}{(1 - \alpha_z)} \left[ \frac{1}{\beta_z} \left( \frac{\overline{XS}_z}{B_z} \right)^{\kappa_z} - \frac{(1 - \beta_z)}{\beta_z} IMT_{zj}^{\kappa_z} \right]^{\frac{-\rho_z}{\kappa_z}} \right\}^{\frac{1}{\rho_z}} \quad [\text{068}]$$

## APPENDIX I: EQUATION [070]

The 2-region P-Model is

$$\frac{EXT_z}{D_z} = \left( \frac{\beta_z}{1 - \beta_z} \frac{PXT_z^*}{PL_z^*} \right)^{\tau_z} \quad [006]$$

$$\frac{IMT_z}{D_z} = \left( \frac{1 - \alpha_z}{\alpha_z} \frac{PL_z^*}{PMT_z^*} \right)^{\sigma_z} \quad [010]$$

$$PXT_{zj}^* = PMT_z^* \quad [069]$$

Substitute from [069] into [006] and develop.

$$\frac{\beta_z}{1 - \beta_z} \frac{PMT_{zj}^*}{PL_z^*} = \left( \frac{EXT_z}{D_z} \right)^{1/\tau_z} \quad [I001]$$

$$\frac{1 - \beta_z}{\beta_z} \frac{PL_z^*}{PMT_{zj}^*} = \left( \frac{D_z}{EXT_z} \right)^{1/\tau_z} \quad [I002]$$

$$PL_z^* = \frac{\beta_z}{1 - \beta_z} \left( \frac{D_z}{EXT_z} \right)^{1/\tau_z} PMT_{zj}^* \quad [I003]$$

Next, develop [010].

$$\frac{1 - \alpha_z}{\alpha_z} \frac{PL_z^*}{PMT_z^*} = \left( \frac{IMT_z}{D_z} \right)^{1/\sigma_z} \quad [I004]$$

$$PL_z^* = \frac{\alpha_z}{1 - \alpha_z} \left( \frac{IMT_z}{D_z} \right)^{1/\sigma_z} PMT_z^* \quad [I005]$$

Combine [I003] and [I005]:

$$\frac{\alpha_z}{1 - \alpha_z} \left( \frac{IMT_z}{D_z} \right)^{1/\sigma_z} PMT_z^* = \frac{\beta_z}{1 - \beta_z} \left( \frac{D_z}{EXT_z} \right)^{1/\tau_z} PMT_{zj}^* \quad [I006]$$

$$PMT_z^* = \frac{1 - \alpha_z}{\alpha_z} \left( \frac{D_z}{IMT_z} \right)^{1/\sigma_z} \frac{\beta_z}{1 - \beta_z} \left( \frac{D_z}{EXT_z} \right)^{1/\tau_z} PMT_{zj}^* \quad [I007]$$

Equation [070] follows directly.

$$\frac{PMT_z^*}{PMT_{zj}^*} = \frac{1 - \alpha_z \left( \frac{D_z}{IMT_z} \right)^{1/\sigma_z}}{\alpha_z} \frac{\beta_z \left( \frac{D_z}{EXT_z} \right)^{1/\tau_z}}{1 - \beta_z} \quad [070]$$