The Dynamics of Poverty: A Review of Decomposition Approaches and Application to Data From Burkina Faso

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Abstract Recent years have been characterized by significant efforts to understand and fight poverty, especially in Africa. Decomposing the dynamics of poverty is one of the focuses of this analysis, which seeks to evaluate the contribution some major factors make to the evolution of the phenomenon of poverty. Several approaches to this decomposition have been proposed in the literature. Our study draws on the Shapley value as a theoretical foundation for various decompositions and reviews the primary methods in existence. These methods are illustrated using data from Burkina Faso. This work was conducted with technical support from CREFA at the Université Laval and provides a framework for advanced training sessions jointly conducted by SISERA and the WBI.

Key Words: Dynamics of Poverty, Shapley Decomposition, Burkina Faso

1 INTRODUCTION

The phenomenon of poverty is very pervasive in Africa. For example, the incidence of poverty rose from 30 per cent in 1985 to 46 per cent in 1988 in Côte d'Ivoire (Grootaert, 1996), from 44.5 per cent in 1994 to 45.3 per cent in 1998 in Burkina Faso (INSD, 1996, 2000), and from 48 per cent in 1994 to 52.9 per cent in 1997 in Kenya (Greda et al., 2001)—demonstrating both the extent of the problem and a worsening trend.

Recent years have been characterized by a significant effort in terms of research aimed at understanding this phenomenon. Moreover, various African countries have elaborated *Poverty Reduction Strategy Papers (PRSP)* under the aegis of the World Bank. Among the methods for fighting poverty, economic growth and income redistribution using a variety of mechanisms occupy a central position. The intertemporal evolution of poverty, and particularly its decomposition into growth, redistribution, and sectorial effects is of vital importance to researchers, donors, and political decision makers.

This document examines this decomposition of poverty and seeks to present the existing methods for decomposing the intertemporal evolution of poverty.

This work is part of the training activities jointly conducted by SISERA, the WBI, and CREFA aimed at familiarizing researchers and African decision-makers with the tools of poverty analysis.

2 REVIEW OF THE LITERATURE

2.1 AN OVERVIEW OF THE DECOMPOSITION ISSUE

Poverty and inequality are usually measured using quantitative indices. For example, when policies are implemented to reduce poverty, it becomes important to measure the evolution of these indices, and especially the decomposition of the observed variation, in order to evaluate the contribution of potential explanatory factors.

A general overview of the decomposition issue is presented by Shorrocks (1999) as follows. Let I be an aggregate indicator representing a poverty or inequality measure, and let X_k, $k = 1,2,\ldots,m$ be a set of factors contributing to the value of I. We can write

$$I = f(X_1, X_2, \ldots X_m), \tag{1}$$

where $f(\cdot)$ is an appropriate aggregation function. The goal of all decomposition techniques is to attribute contributions, C_k, to each of the factors, X_k, so that, ideally, the value of I will be equal to the sum of the m contributions.

Each of the decomposition techniques, whether static or dynamic, yields a particular solution to this general decomposition problem as a function of the characteristics of I and the goals of the decomposition. To illustrate, we present several examples of those most frequently used. In the static decomposition of the FGT (P_α) indices proposed by Foster et al. (1984), I is incorporated into P_α, and the factors X_k are population subgroups. In the dynamic poverty decomposition proposed by Datt and Ravallion (1992), I is assimilated into the variation of P_α between two dates, and the variables X_k are variations in growth and redistribution. Other examples of decompositions are found in Kakwani (1993, 1997) for poverty, and in Fields and Yoo (2000), Shorrocks (1982), and Chantreuil and Trannoy (1999) for inequality.

Shorrocks (1999) emphasizes that decomposition techniques confront four principal problems:

1. The contribution assigned to each specific factor does not always have an intuitively clear meaning.

2. Decomposition procedures are only applicable to certain poverty and inequality indices. When used with other indices, these decomposition techniques sometimes introduce vague notions, such as "residual" or "interaction," to ensure the identity of the decomposition.

3. The types of contributing factors considered are usually limited. For example, a single criterion is used to divide the population into subgroups. When multiple

criteria are used for the subdivision, the decomposition methods have difficulty identifying the contributions.

4. All of these decomposition methods lack a shared theoretical framework. Each individual application is viewed as a different problem requiring a different solution.

To introduce a unified theoretical framework, Shorrocks (1999) relies on the Shapley value (cf. Section 2.2) and demonstrates that this approach allows of most of the results of the decomposition to be derived. We present this unified framework and apply it to several techniques for decomposing poverty over time.

2.2 A THEORETICAL FRAMEWORK BASED ON THE SHAPLEY VALUE

2.2.1 DEFINITION OF THE SHAPLEY VALUE

The Shapley value is a solution concept widely used in the theory of cooperative games (Owen, 1977; Moulin, 1968; Shorrocks, 1999). Consider a set, *N*, comprising *n* players who must allocate a gain or loss between themselves. To accomplish this division, the players may form coalitions, i.e. subsets, *S*, of *N*. The strength of each coalition is expressed as a *characteristic function*, *v*. For a given coalition *S*, $v(S)$ measure the share of the surplus that *S* is able to appropriate without resorting to agreements with players belonging to other coalitions. The question to be answered is: How should the surplus be split between the *n* players? Various solutions have been proposed, including that of Lloyd Shapley in 1953.

For each player, *i*, Shapley (1953) proposes a value based on his marginal contribution—defined as the weighted mean of the marginal contributions $v(S \cup \{i\}) - v(S)$ of player *i* in all coalitions $S \subset N - \{i\}$. To delimit this value, we consider all *n* players to be randomly ranked in some order, $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, and then successively eliminated in that order. The elimination of players reduces the share accruing to the group of those not yet eliminated. When the coalition, *S*, is composed of *s* elements, we can only find the value they will obtain, $v(S)$, when the *s* first elements of σ are exactly the elements of *S*.

The weight of the coalition, *S*, will be measured by the probability that the first *s* elements of σ are all elements of *S*. This probability is found by dividing the number of orders of which the first *s* elements are all in *S* by the total number of possible orders. The number of possible orders is the number of permutations of *n* players taken *n* at a time, yielding *n*! (see the Appendix). Similarly, since the first *s* players yield *s*! permutations, the last *n*–*s*–1 players yield $(n-s-1)!$ permutations. The number of orders in which the first *s* players are all elements of *S* is thus given by $s!(n-s-1)!$. (Cf. fundamental principles of combinatorial analysis in the Appendix).

The weight is thus defined by $s!(n-s-1)!/n!$, where *s* is the size of the coalition *S*. This weight also measures the probability that the player before player *i* will be in *S*. The Shapley value for player *i* is thus:

$$\phi_i = \sum_{0\le s\le n-1} \frac{s!(n-s-1)!}{n!} \sum_{\substack{S\subset N-\{i\} \\ |S|=s}} \left[v(S\cup\{i\}) - v(S)\right], \tag{2}$$

where, by convention, $0!=1$ and $v(\varnothing)=0$.

A detailed description of the Shapley value is given in Moulin (1988, Chapter 5). This value provides the framework for several types of decomposition. For example, Chantreuil and Trannoy (1999) use it to decompose inequality by income source. Shorrocks (1999) generalizes application of the decomposition to any index *I* defined in equation 1.

2.2.2 APPLICATION OF THE SHAPLEY VALUE TO DECOMPOSING POVERTY

Shorrocks' (1999) general procedure consists of estimating the marginal effect on *I* of removing each contributing factor in a given elimination sequence. Repeating the operation for all possible elimination sequences, we compute the mean of the marginal effects for each factor. This mean measures the contribution of the chosen factor, yielding an exact, additive decomposition of *I* into *m* contributions. This approach is formalized in the following paragraphs. In contrast to the notation in the preceding presentation, we now deal with *m* factors instead of *n* players, but the procedure is the same.

Consider a poverty or inequality measure *I* defined in equation 1, the value of which is completely determined by a set of *m* contributing factors X_k, where $k \in K = \{1,2,\ldots,m\}$. *I* may be a static measure of poverty or inequality, or it may represent their variation over time. In this paper we are interested in intertemporal variations in poverty. As previously indicated, contributions are determined by a sequential elimination procedure. The *m* factors are ranked in some order of elimination. The act of eliminating some elements causes subsets, or coalitions, *S*, to appear. We call $F(S)$ the value assumed by *I* when the factors $X_k, k \notin S$ are eliminated. In other words, $F(S)$ is the value assumed by *I* when only the subset of factors *S* is considered (i.e. factors that have not been eliminated).

The structure of the model will be characterized by $\langle K,F \rangle$, i.e. a set of *K* factors and a function $F:\{S | S \subseteq K\} \rightarrow 3$. Since the value of I is entirely determined by the *K* variables, *I* will be equal to zero (0) when all the variables are eliminated, which is tantamount to writing $F(\varnothing)=0$. The decomposition of $\langle K,F \rangle$ yields real values for $C_k, k \in K$. C_k measures the contribution of each factor, *k*, and can be written:

$$C_k = C_k(K,F), \quad k \in K. \tag{3}$$

Two properties are required of this decomposition. The first is symmetry, ensuring that the contribution of each factor is independent of the order in which it appears in the list or sequence. The second property is exactness and additivity, which can be written:

$$\sum_{k \in K} C_k(K,F) = F(K), \quad \forall \langle K,F \rangle. \tag{4}$$

When the additivity condition holdsx (equation 4), $C_k(K,F)$ can be interpreted as the contribution of factor *k* to the inequality or poverty measured by *I*. Similarly, it should also be possible to interpret the contribution of each factor *k* as its marginal impact, yielding:

$$M_k(K,F) = F(K) - F(K - \{k\}), \quad k \in K. \tag{5}$$

If the condition, or rule, expressed in equation 5 obtains, the decomposition is symmetric, though not necessarily exact. The marginal effect may also be estimated if the factors are eliminated sequentially. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ be the order in which factors are eliminated, and

$S(\sigma_r,\sigma)=\{\sigma_i | i>r\}$ all factors remaining after the factor σ is eliminated. The marginal effects are given by:

$$C_k^{\sigma}=F[S(k,\sigma)\cup\{k\}]-F[s(k,\sigma)]=\Delta_k F[S(k,\sigma)],\quad k\in K, \tag{6}$$

where $\Delta_k F(S)=F(S\cup\{k\})-F(S)$, with $S\subseteq K-\{k\}$, is the marginal impact of adding factor *k* to the set *S*. Given that $S(\sigma_r,\sigma)=S(\sigma_{r+1},\sigma)\cup\{\sigma_{r+1}\}$, $r=1,2,\ldots,m-1$, we conclude that:

$$\begin{aligned}\sum_{k\in K}C_k^{\sigma}&=\sum_{r=1}^{m}C_{\sigma_r}^{\sigma}=\sum_{r=1}^{m}\left(F[s(\sigma_r,\sigma)\cup\{\sigma_r\}]-F[S(\sigma_r,\sigma)]\right),\\&=F[S(\sigma_{1r},\sigma)\cup\{\sigma_1\}]-F[S(\sigma_m,\sigma)]=F(K)-F(\varnothing)=F(K).\end{aligned} \tag{7}$$

Equation 7 yields the exact value of $F(K)$, since $F(\varnothing)=0$. In this equation, each factor's contribution depends upon its rank in the list, i.e. the elimination path. However, the global value $F(K)$ is the same regardless of the permutation of the factors. To solve the problem of the ordering playing a role, and to ensure a symmetric decomposition, we take all possible elimination sequences, i.e. a total of *m*! sequences $\sigma\in\Omega$, and compute the expected value of C_k^{σ} when the sequences in Ω are chosen at random. The following decomposition of C^S results:

$$C_k^S(K,F)=\frac{1}{m!}\sum_{\sigma\in\Omega}C_k^{\sigma}=\frac{1}{m!}\sum_{\sigma\in\Omega}\Delta F[S(k,\sigma)]=\sum_{s=0}^{m-1}\sum_{\substack{S\subseteq K-\{k\}\\|S|=s}}\frac{(m-1-s)!\,s!}{m!}\Delta F(S). \tag{8}$$

This decomposition C^S (in equation 8) is exact, additive, and also symmetric. The last term corresponds to the Shapley value defined in equation 2. We shall refer to this relationship as the Shapley decomposition rule (Shorrocks, 1999). The contribution of each factor, *k*, may be interpreted as its expected marginal impact when all possible elimination paths are considered. The factorial $[(m-1-s)!\,s!/m!]$, also denoted $\pi(s,m-1)$ by Shorrocks (1999), is a weight defined in section 2.2.1. It gives the probability of choosing the subset *S*, of size *s*, in a large set *M* with *m*–1 elements when each subset whose size is between 0 to *m–1* has the same likelihood. In the remainder of the text, we use the simplified expression $C_k^S(K,F)=\underset{S\subseteq K-\{k\}}{\varepsilon}\Delta_k F(S),\ k\in K$ to designate the Shapley value, or the contribution of factor *k*.

3 APPROACHES TO DECOMPOSING INTERTEMPORAL VARIATIONS IN POVERTY

3.1 THE SHAPLEY DECOMPOSITION

The Shapley value can be applied to various categories of poverty or inequality decomposition. We concentrate on the decomposition of the intertemporal evolution of poverty by applying the Shapley approach to two types of decomposition: (1) the decomposition of variations in poverty into a "growth" effect and a "redistribution" effect, and (2) the decomposition of the variation in poverty into sectorial effects by population subgroup.

3.1.1 THE CONTRIBUTION OF GROWTH AND REDISTRIBUTION

Intertemporal movements in poverty are assumed to be explained by two factors, income growth and distribution shifts. Given a fixed poverty line, the level of poverty at time t; $(t = 1,2)$ may be expressed by a function $P(\mu_t, L_t)$ of mean income, μ_t, and the Lorenz curve, L_t. The growth factor is $G = \mu_2/\mu_1 - 1$, and the redistribution factor $R = L_2 - L_1$.

The decomposition issue here consists of identifying the contribution of growth, *G*, and that of redistribution, *R*, to the variation in poverty, ΔP. Comparing this particular decomposition problem with the general formulation expressed in equation 1 (section 2.1), we observe that ΔP is integrated into *I*, while the variables in X_k are *G* and *R*. Consequently, we can write:

$$\Delta P = P(\mu_2, L_2) - P(\mu_1, L_1) = P[\mu_1(1+G), L_1 + R] - P(\mu_1, L_1) = F(G, R). \tag{9}$$

The contribution of *G* and *R* to the variation in poverty, ΔP, is computed from the Shapley value expressed in equation 8 (section 2.2).

Since there are two factors, i.e. $m = 2$, we have two $(m! = 2! = 2)$ possible elimination sequences. They are:

Sequence A: $\sigma_A = \{G, R\}$

Sequence B: $\sigma_B = \{R, G\}$

The contribution of growth can be expressed as

$$C_G^S = \frac{1}{2}\left\{\underbrace{\Delta_G F[S(G,\sigma_A)]}_{\text{Sequence A}} + \underbrace{\Delta_G F[S(G,\sigma_B)]}_{\text{Sequence B}}\right\}. \tag{10}$$

Here, in equation 10, the first addend, capturing the sequence A, is given by the value $F[S(G,\sigma_A)\cup\{G\}]-F[S(G,\sigma_A)]=F(G,R)-F(R)$. Indeed, $S(G,\sigma_A)$ indicates that all elements up to *G* have been eliminated from the sequence A (only *R* remains), and if we reintroduce *G* with the union of $\{G\}$, we obtain the pair (G,R).

The second addend, capturing the sequence B, is developed similarly, yielding the value $F[S(G,\sigma_B)\cup\{G\}]-F[S(G,\sigma_B)]=F(G)-F(\varnothing)$. In this case, $S(G,\sigma_B)$ also indicates that all elements up to *G* have been eliminated from the sequence B (there are no more elements). Then, if we bring back *G* through introducing the union with $\{G\}$, we obtain the only element, *G*.

Finally, $C_G^S = \frac{1}{2}[F(G,R)-F(R)+F(G)-F(\varnothing)] = \frac{1}{2}[F(G,R)-F(R)+F(G)]$. From equation 9 we can obtain a final expression for the contribution of growth:

$$\begin{aligned} C_G^S &= \frac{1}{2}[F(G,R)-F(R)+F(G)], \\ &= \frac{1}{2}\{P(\mu_2,L_2)-P(\mu_1,L_1)-[P(\mu_1,L_2)-P(\mu_1,L_1)]+[P(\mu_2,L_1)-P(\mu_1,L_1)]\}, \\ &= \frac{1}{2}\{P(\mu_2,L_2)-P(\mu_1,L_2)+[P(\mu_2,L_1)-P(\mu_1 L_1)]\}. \end{aligned} \tag{11}$$

This expression, equation set 11, reveals that, according to the Shapley rule, the contribution of the "growth" factor is equal to the mean of two elements: (1) the variation in the poverty measure if inequality is fixed at its value in the first period, and (2) the variation in the poverty measure if inequality is fixed at its value in the last period.

Considering the same sequences A and B defined above, the contribution of inequality is defined similarly: $C_G^S = \frac{1}{2}[F(R)-F(\varnothing)+F(G,R)-F(G)] = \frac{1}{2}[F(G,R)-F(G)+F(R)]$.

$$\begin{aligned} C_G^S &= \frac{1}{2}[F(G,R)-F(G)+F(R)], \\ &= \frac{1}{2}\{P(\mu_2,L_2)-P(\mu_1,L_1)-[P(\mu_2,L_1)-P(\mu_1,L_1)]+[P(\mu_1,L_2)-P(\mu_1,L_1)]\}, \\ &= \frac{1}{2}\{P(\mu_2,L_2)-P(\mu_2,L_1)+[P(\mu_1,L_2)-P(\mu_1 L_1)]\}. \end{aligned} \tag{12}$$

This expression, equation set 12, shows the contribution of the "inequality" factor according to the Shapley rule. It is equal to the mean of two elements: (1) the variation in the poverty measure if mean income is fixed at its value in the first period, and (2) the variation in the poverty measure if mean income is fixed at its value in the last period.

Finally, the variation in poverty is $\Delta P = C_G^S + C_R^S$, i.e. the sum of the contributions of growth and distribution.

3.1.2 SECTORIAL DECOMPOSITION OF VARIATIONS IN POVERTY

The population whose poverty is under study may be subdivided into several subgroups or socio-economic sectors. It is frequently of some interest to evaluate the contribution of each subgroup to the variation in poverty between two periods. We present the application of the Shapley rule to this type of decomposition, as presented by Shorrocks (1999).

Let *K* be the set of all subgroups and P_t global poverty in the population in period *t*. Furthermore, let α_{kt} be the proportion in the overall population of the group $k \in K$, and P_{kt} its FGT poverty measure at time $t; (t = 1,2)$. The decomposability property of FGT indices allows us to write $P_t = \sum_k \alpha_{kt} P_{kl}$. The variation in poverty between the two periods is $\Delta P = \sum_k (\alpha_{k2} P_{k2} - \alpha_{k1} P_{k1})$ and is contingent on the shares, $\Delta \alpha_k$, and on the poverty measures, ΔP_k, within each group.

Shorrocks (1999) shows that the Shapley decomposition of ΔP into the contributions of share and poverty variations is given by the relationship:

$$\Delta P = \sum_{k \in K} \frac{\alpha_{k1} + \alpha_{k2}}{2} \Delta P_k + \sum_{k \in K} \frac{P_{k1} + P_{k2}}{2} \Delta \alpha_k \ . \qquad (13)$$

The first sum is the contribution of each groups' poverty variations, and the second the contribution of changes in the population shares. Since it is additive, the contribution of a given sector, *k*, is: $C_k = (\alpha_{k1} + \alpha_{k2}) \frac{\Delta P_k}{2} + (P_{k1} + P_{k2}) \frac{\Delta \alpha_k}{2}$. We can easily verify that C_k arises from applying the Shapley value to the decomposition of the variation of an index between two factors (see equations 11 and 12).

Furthermore, other sectorial decompositions are found in the literature. The most widely used is presented by Ravallion and Huppi (1991), and Ravallion (1996). It also makes use of the additivity property of the FGT class of poverty measures.

Treating the first period as the base year, and adopting the same definitions as above, the poverty variation between two dates, $t = 1,2$, is decomposed as:

$$\begin{aligned} \Delta P &= \sum_k (P_{k2} - P_{k1})\alpha_{k1} && \text{Intra - sectorial effects} \\ &+ \sum_k (\alpha_{k2} - \alpha_{k1})P_{k1} && \text{Population shift effects} \\ &+ \sum_k (P_{k2} - P_{k1})(\alpha_{k2} - \alpha_{k1}) && \text{Interaction effects.} \end{aligned}$$

Intra-sectorial effects represent the contribution of each sectors' poverty when we freeze the population shares at their base-period levels.

Population shift effects indicate to what extent initial poverty (base-year period) is reduced by various shifts in the population proportions in each sector between the two dates (1 and 2).

Interaction effects spring from a potential correlation between sectorial gains and population shifts. Their signs reveal whether the population tends to shift towards sectors in which poverty is falling.

The sectorial decomposition programmed into the DAD software uses this decomposition procedure, not the Shapley decomposition. Nonetheless, the DAD allows calculation of the parameters necessary for applying the Shapley approach formulated in equation 13.

3.2 The Standard Datt and Ravallion Approach

Datt and Ravallion (1992) proposed a decomposition of poverty variations allowing an evaluation of the contributions of growth and inequality. Variations in poverty are thus decomposed into three elements: (1) a growth effect, measuring changes in poverty that would be obtained if the Lorenz curve remained stable, (2) a redistribution effect, evaluating the changes in poverty imputable to a shift in the Lorenz curve when mean income is constant, and (3) a residual measuring the interaction between the growth and redistribution effects.

Given a reference period, *r*, the variation in poverty between periods *t* and *t*+1 is decomposed as follows:

$$P_{t+1} - P_t = \underbrace{G(t, t+1, r)}_{\text{Contribution of growth}} + \underbrace{D(t, t+1, r)}_{\text{Contribution of inequality}} + \underbrace{R(r, t+1, r)}_{\text{Residual}}, \tag{17}$$

where:

$$G(t, t+1, r) = P\left(\frac{z}{\mu_{t+1}}, L_r\right) - P\left(\frac{z}{\mu_t}, L_r\right), \tag{18}$$

$$D(t, t+1, r) = P\left(\frac{z}{\mu_r}, L_{t+1}\right) - P\left(\frac{z}{\mu_r}, L_t\right), \tag{19}$$

$$R(t, t+1, t) = G(t, t+1, t+1) - G(t, t+1, t) = D(t, t+1, t+1) + D(t, t+1, t). \tag{20}$$

This final "residual" does not exist in the Shapley decomposition (section 3.1), nor in Kakwani's (1997).

3.3 THE SHAPLEY-OWEN-SHORROCKS APPROACH (SOS)

In the presentation of the general issue of decomposition in section 2.1, the variables X_i were assumed to be individual units, not composites. In reality, X_i may be a primary unit composed of a number of secondary variables. Similarly, we may wish to group some variables X_k into a larger group. These cases lead to a hierarchical model in which variables, or secondary factors, are grouped into so-called primary units.

Application of the Shapley decomposition independently to the set of secondary units, then to the set of primary units, does not guarantee that the contribution of a given primary unit will equal the sum of the contributions of its constituent secondary units. To ensure consistency of the decomposition, Shorrocks (1999) proposes a two-stage Shapley-style decomposition that draws on an approach developed by Owen (1977). The procedure thus obtained was called the "Shapley-Owen-Shorrocks" procedure, abbreviated to SOS, by Shorrocks and Kolenilov (2001). The first step decomposes the poverty or inequality measure, *I*, to estimate the contributions of the primary units, and the second step decomposes the contribution of each primary unit into as many contributions as there are secondary subunits.

The detailed structure of the model is written $\langle K, F\rangle$ and includes *K* secondary factors. We assume that the *K* secondary factors are grouped into *A* primary factors, yielding the hierarchical model $\langle K, A, F\rangle$, with $A=\{L_j,\ j\in J\}$. We denote $C_k^*(K, A, F)$ the contribution of each secondary factor, $k\in K$, and $C_L^*(K, A, F)$ that of each primary factor $L\in A$.

The aggregation is called consistent if the contribution of each primary factor is the sum of the contributions of its constituents, which can be written:

$$C_L^*(K, A, F)=\sum_{k\in L} C_k^*(K, A, F), \quad \text{for each } L\in A. \tag{25}$$

Replacing each secondary factor by the corresponding primary factory, we obtain an aggregate model $\langle A, F^A\rangle$. When the Shapley approach is applied individually to the detailed model $\langle K, F\rangle$ and to the aggregate model $\langle A, F^A\rangle$, consistency is not assured—thence the SOS approach.

The SOS method is a two-stage sequential procedure when the factors are grouped into two (2) hierarchical levels (primary and secondary factors). When there are more levels, the SOS method can be applied in several steps.

In the two-stage case, the first step determines the contributions of the primary factors. Using the aggregate model, $\langle A, F^A\rangle$, and applying the Shapley value given in equation 8, the contribution of each primary factor *L* is given by:

$$C_L^S(A, F^A)=\underset{T\subseteq A-\{L\}}{\varepsilon}\Delta_L F^A(T)=\underset{T\subseteq A-\{L\}}{\varepsilon}\left[F^A(T\cup\{L\})-F^A(T)\right]=\overline{F}_L(L), \quad L\in A. \tag{26}$$

In the second step, the contribution of each primary factor *L*, $\overline{F}_L(L)$, is allocated to the constituents, $(k\in L)$, by applying the Shapley decomposition to the model $\langle L, \overline{F}_L\rangle$. The contribution of each secondary factor, *k*, within its group, *L*, is given by:

$$C_k^S(L, \overline{F}_L)=\underset{S\subseteq L-\{k\}}{\varepsilon}\Delta_k \overline{F}_L(S), \quad k\in L. \tag{27}$$

An important theoretical condition for consistency of the decompositions is that the function *F* be separable into its constituents. In other words, the marginal contribution of each factor, $k \in L$, does not depend on that of any other factor in *L*. If the contributions of some factors are interrelated, they should be treated as a single entity. The function *F* should also be separable for each $L \in A$. The specifics of this separability of *F* can be found in Shorrocks (1999). This condition also corresponds to that of a cooperative game with transferable utility, details of which can be found in Moulin (1988).