SISERA TRAINING WORKSHOP
ON POVERTY DYNAMICS

The Shapley Value

By
Araar Abdelkrim

Kampala, Uganda
January 22-30, 2003

THE SHAPLEY APPROACH

The Shapley value is a solution concept often employed in the theory of cooperative games. We consider a set N, constituted of n players that must divide a given surplus among themselves. The players group themselves together to form coalitions (these are the subsets S of N) that appropriate themselves a part of the surplus and redistribute it between their members. We suppose that the function v determines the coalition force, i.e., which surplus will be divided without resorting to an agreement with the other players (that are members of the complementary coalition "(n-s-1) players"). The question to resolve is the following one: how can we divide the surplus between the n players? According to the Shapley approach, introduced by Lloyd Shapley in 1953, the value of the player k in the game, given by the characteristic function v, is given by the following formula:

$$C_k = \sum_{\substack{s \subset S \\ s \in \{0, n-1\}}} \frac{s!(n-s-1)!}{n!} MV(S,k) \quad \text{And} \quad MV(S,k) = \left(v\left(S \cup \{k\}\right) - v(S)\right) \tag{01}$$

EXPLANATION AND INTERPRETATION OF THE SHAPLEY VALUE

The term, MV (S, k) is equal to the marginal value that the player k generates after his adhesion to the coalition S. What will then be the expected marginal contribution of player k, according to the different possible coalitions that can be formed and to which the player can adhere? First of all, the size of the coalition S is limited such as: $s \in \{0,1,\ldots n-1\}$. Suppose that the n players are randomly ordered according to an order, noted by σ, such as:

$$\sigma = \left\{ \underbrace{\sigma_{1,}\sigma_2,\cdots,\sigma_{k-1}}_{s}, \sigma_k, \underbrace{\sigma_{k+1},\cdots,\sigma_n}_{n-s-1} \right\} \tag{02}$$

For each of the possible permutations of the n players, which number n!, the number of times that the same first s players are located in the subset or coalition S is given by the number of possible permutation of the s players in coalition S, that is, s!.

For every permutation in the coalition S, we find (n-s-1)! permutations for the players that complement the coalition S.

The expected marginal value that player k generates after his adhesion to a coalition S is given by the Shapley value, as indicated in equation (01).

For every position of the factor k (predetermined cuts of the coalition S), there are several possibilities to form coalitions S from the n-1 player (the n players without the player k). This number of possibilities is equal to the combination C_{n-1}^{s}.

How many marginal values would one have to calculate to calculate the expected marginal contribution of a given factor, be the factor k?

Because the order of the players in the coalition S does not affect the contribution of the player k once he has adhered to the coalition, the number of calculations needed for the marginal values is reduced from n! to $\sum_{s=0}^{n-1} C_{n-1}^{s} = 2^{n-1}$ (see annex A).

Remark:
When the marginal contribution of the factor k, MV(S, k) is same whatever be the order or the coalition S, the Shapley value for the factor k will then be given by this that follows:

$$C_k = \sum_{\substack{s \subset S \\ s \in \{0, n-1\}}} \frac{s!(n-s-1)!}{n!} \overline{x} \qquad \textbf{(03)}$$

$$C_k = C_{n-1}^{s} \sum_{s=0}^{n-1} \frac{s!(n-s-1)!}{n!} \overline{x} = \frac{n!}{s!(n-s)!} \frac{ns!(n-s-1)!}{n!} \overline{x} = \overline{x} \qquad \textbf{(04)}$$

QED

ILLUSTRATION OF THE SHAPLEY VALUE WITH TWO FACTORS (2 PLAYERS)

Suppose an index I that depends on two factors A and B. The decomposition of this index consists in calculating the expected marginal contribution of every factor.

Number of permutations = $n! = 2! = 2$

	Permutations	**Marginal contribution of factor A**	
1	$\sigma^1 = \{A, B\}$ $S = \{\not A, \not B\} = \{\phi\}$ $S \cup \{A\} = \{A\}$ $s = 0$	$\left(v\left(S \cup \{k\}\right) - v(S)\right) = I(A) - 0$	In this first case, one looks for the marginal contribution of the added factor A to a set of factors that is empty.
2	$\sigma^2 = \{B, A\}$ $S = \{B, \not A\} = \{B\}$ $S \cup \{A\} = \{B, A\}$ $s = 1$	$\left(v\left(S \cup \{k\}\right) - v(S)\right) = I(A, B) - I(B)$	Here, the set of factors (coalition) that already exist is the factor B.

Concordances with the Shapley formula:

Possible coalitions for the factor A	S	$C = N - \left(S \cup \{A\}\right)$	Weight attributed to the factor A according to the case
Case 1	$S = \{\phi\},\ s = 0$	$C = \{B\},\ n - s - 1 = 1$	$s!\left(n - s - 1\right)! / n! = 1/2$
Case 2	$S = \{B\},\ s = 1$	$C = \{\phi\},\ n - s - 1 = 0$	$s!\left(n - s - 1\right)! / n! = 1/2$

$$C_A = \frac{1}{2} I(A) + \frac{1}{2}\left(I(A, B) - I(B)\right) = \frac{1}{2}\left[I(A, B) - I(B) + I(A)\right] \qquad \textbf{(05)}$$

$$C_B = \frac{1}{2} I(B) + \frac{1}{2}\left(I(A, B) - I(A)\right) = \frac{1}{2}\left[I(A, B) - I(A) + I(B)\right] \qquad \textbf{(06)}$$

ILLUSTRATION OF THE SHAPLEY VALUE WITCH THREE FACTORS (3 PLAYERS)

Suppose an index I that depends on three factors A, B and C. The decomposition of this index consists to calculate the expected marginal contribution of every factor.

Number of permutation = $n! = 3! = 6$

case	Permutation	Size of coalition S	Marginal contribution of factor A
1	$\sigma^1 = \{A, B, C\}$	s=0	I(A)-0
2	$\sigma^2 = \{A, C, B\}$	s=0	I(A)-0
3	$\sigma^3 = \{B, A, C\}$	s=1	I(B,A)-I(B)
4	$\sigma^4 = \{B, C, A\}$	s=2	I(B,C,A)-I(B,C)
5	$\sigma^5 = \{C, A, B\}$	s=1	I(C,A)-I(C)
6	$\sigma^6 = \{C, B, A\}$	s=2	I(C,B,A)-I(C,B)

If we note the marginal variation, that results from the elimination of the factor k for the sequence I, by: $VI(\sigma^i, k)$ (example: $VI(\sigma^1, A) = I(A) - I(\phi)$), the expected marginal contribution of factor k, according the Shapley approach is given by the following equation:

$$C_k = \frac{1}{n!}\sum_{i=1}^{n!} VI(\sigma^i, k) \qquad \textbf{(07)}$$

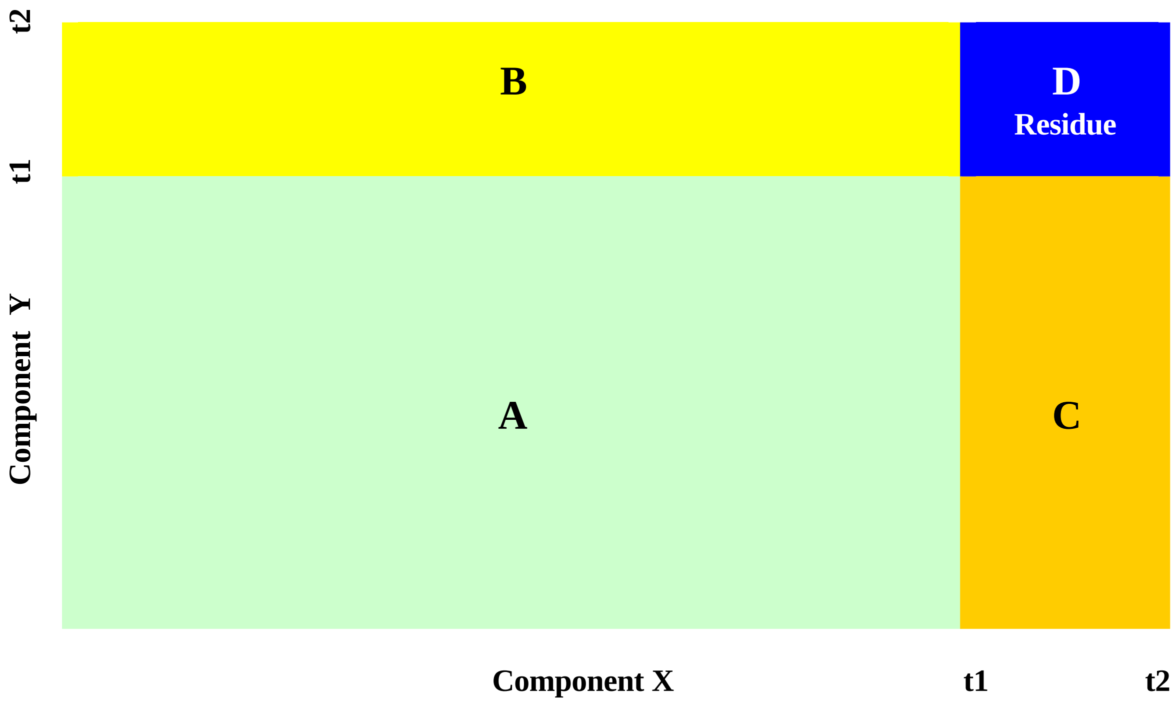
Concordances with the Shapley formula:

Possible coalitions for the factor A	S	N-S-{A}	Weight attributed to the marginal contribution according to the case
Cases 1 and 2 Case 3	$\{\phi\}$ $s = 0$ $\{B\}$ $s = 1$	$\{B, C\}$ $n - s - 1 = 2$ $\{C\}$ $n - s - 1 = 1$	$s!(n - s - 1)!/n! = 2/6$ $s!(n - s - 1)!/n! = 1/6$
Cases 4 and 6	$\{B, C\}$ $s = 2$	$\{\phi\}$ $n - s - 1 = 0$	$s!(n - s - 1)!/n! = 2/6$
Case 5	$\{C\}$ $s = 1$	$\{B\}$ $n - s - 1 = 1$	$s!(n - s - 1)!/n! = 1/6$

$$C_k = \sum_{\substack{s \subset S \\ s \in \{0, n-1\}}} \frac{s!(n - s - 1)!}{n!} MV(S, k) \qquad \textbf{(08)}$$

Geometrical illustration of the Shapley value

Euclidian example



An easy example of an index whose values depend on the interaction between the two components (C_X, C_Y) is the surface of rectangle:

$$I_S(t) = C_X^t * C_Y^t$$

In period t1, I_S=A and in period t2, I_S=A+B+C+D

Suppose that the index VI_S indicates the variation in the index I_S such us:

$$VI_S(t1, t2) = I_S(t2) - I_S(t1)$$

Q1: If the period t1 is the reference period, decompose the index of variation VI_S, such as:

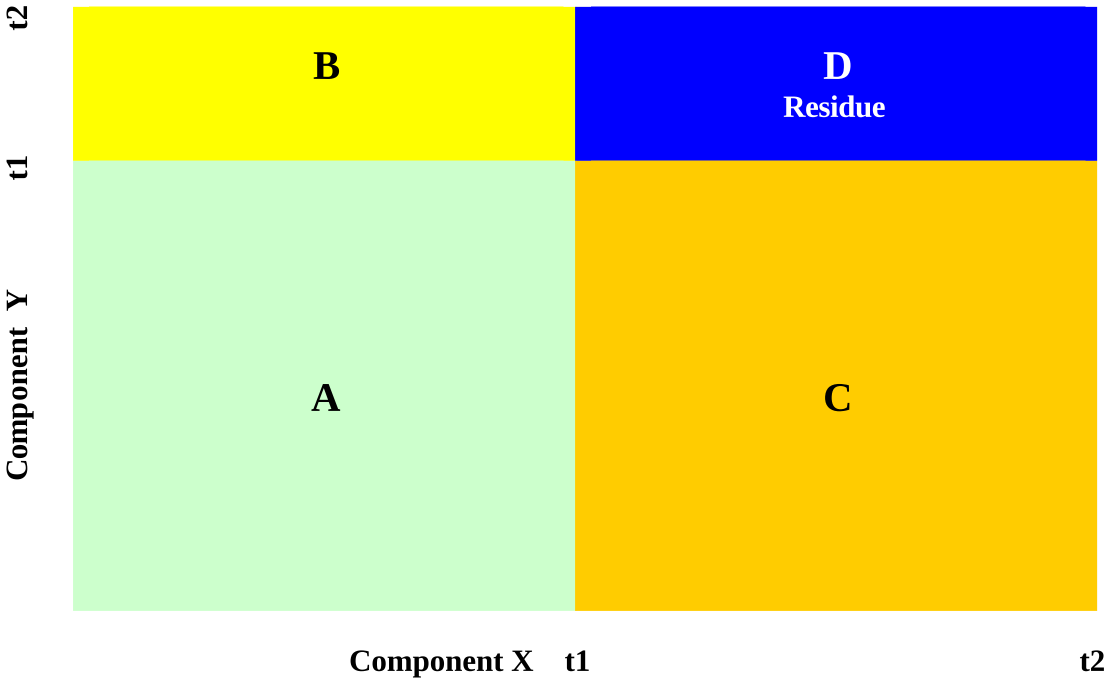
$$VI_S(t1, t2) = \text{Contribution of } X + \text{Contribution of } Y + \text{The residue}$$

Answer: VI_S= C + B +D

Q2: Do the same decomposition of Q1 when the period t2 is the reference period.

Answer: VI_S= (C+D) + (B +D)+(-D)

Q3: By observing this rectangle, in what cases the residue can be important?



Answer: When the ratio of variation of two components, $\Delta X / \Delta Y$ or its inverse is higher.

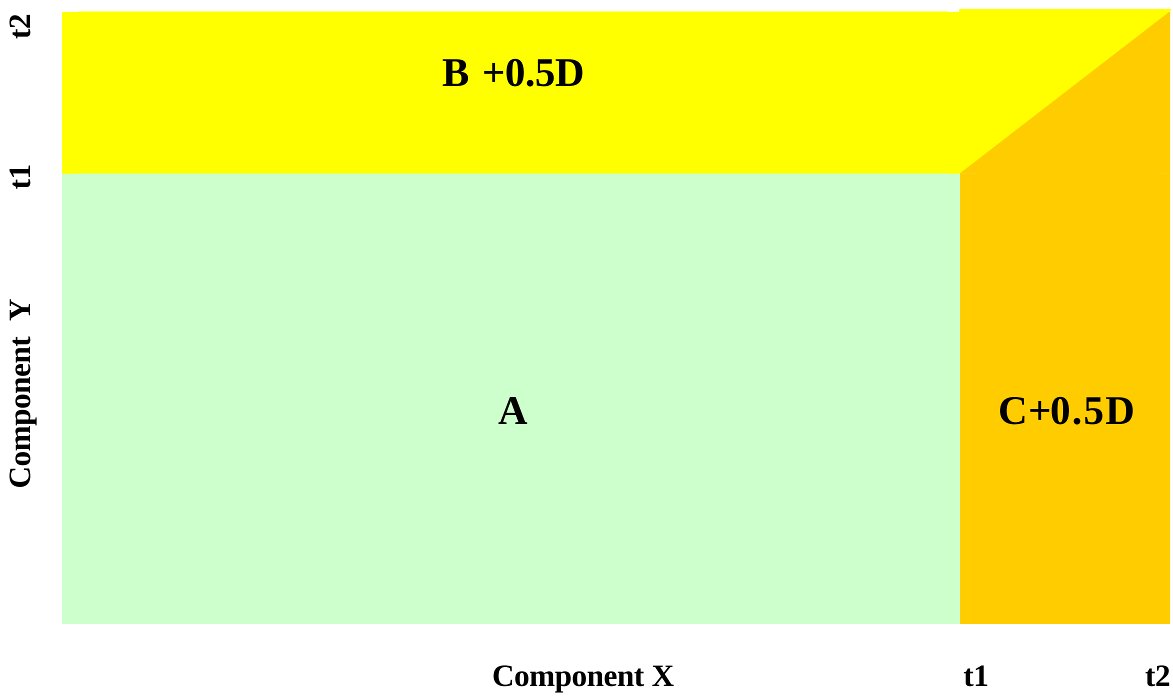
Q4: By using the Shapley approach, propose an exact decomposition of the index of variation of the surface of rectangle VI_S.

$$\text{Contribution of } X = \frac{1}{2}\left(\left[I(C_X^{t2}, C_Y^{t1}) - I(C_X^{t1}, C_Y^{t1})\right] + \left[I(C_X^{t2}, C_Y^{t2}) - I(C_X^{t1}, C_Y^{t2})\right]\right)$$

$$\text{Contribution of } X = \frac{1}{2}\left(\left[(A + C) - (A)\right] + \left[(A + B + C + D) - (A + B)\right]\right) = C + \tfrac{1}{2}D$$

$$\text{Contribution of } Y = \frac{1}{2}\left(\left[I(C_X^{t1}, C_Y^{t2}) - I(C_X^{t1}, C_Y^{t1})\right] + \left[I(C_X^{t2}, C_Y^{t2}) - I(C_X^{t2}, C_Y^{t1})\right]\right)$$

$$\text{Contribution of } Y = \frac{1}{2}\left(\left[(A + B) - (A)\right] + \left[(A + B + C + D) - (A + C)\right]\right) = B + \tfrac{1}{2}D$$



Exercise 2)

By using the Shapley approach, propose an exact sectoral decomposition for the FGT index when the two considered factors are respectively the proportion of the group and the poverty within the group.

Answer:

Contribution of $\varphi_K = \frac{1}{2}\left[\left[\left(\varphi_2(k)P_1(k;\alpha,z)\right)-\left(\varphi_1(k)P_1(k;\alpha,z)\right)\right]+\left[\left(\varphi_2(k)P_2(k;\alpha,z)\right)-\left(\varphi_1(k)P_2(k;\alpha,z)\right)\right]\right]$

Contribution of $P_K = \frac{1}{2}\left[\left[\left(\varphi_1(k)P_2(k;\alpha,z)\right)-\left(\varphi_1(k)P_1(k;\alpha,z)\right)\right]+\left[\left(\varphi_2(k)P_2(k;\alpha,z)\right)-\left(\varphi_2(k)P_1(k;\alpha,z)\right)\right]\right]$

Annex A: Binomial Theorem of Newton.

What Newton discovered was a formula for $(a+b)^n$ that would work for all values of n, including fractions and negatives:

$$(a+b)^n = a^n + na^{n-1}b + [n(n-1)a^{n-2}b^2] / 2! + [n(n-1)(n-2)a^{n-3}b^3] / 3! + \ldots + b^n$$

$$(a+b)^n = \sum_{s=0}^{n} C_n^s a^{n-1} b^n \quad \forall (a,b) \in \Re, n \in N$$

Raising (a+b) to the power n is equivalent to multiplying n identical binomials (a+b). The result is a sum where every element is the product of n factors of type a or b. The terms are thus of the form $a^{n-p}b^p$. Each of these terms is obtained a number of times equal to C_n^p, which is how many times we can choose p elements among n.

When a=b=1, we will have: $(1+1)^n = \sum_{s=0}^{n} C_n^s = 2^n$

Hence, we can conclude that: $\sum_{s=0}^{n} C_{n-1}^s = 2^{n-1}$